1. Consider lattice paths of length n, starting at the origin and ending at (x, y), and using steps N, E, S, W where $S = [0, -1]$ and $W = [-1, 0]$. Let $r = (n - x - y)/2$ and $s = (n + x - y)/2$.

   - Show that the number of such paths is given by

     $$\binom{n}{r}\binom{n}{s}$$

     Solutions: r is the number of S or W steps and s is the number of W or N steps.

   - Show that the number of such paths staying weakly above the x-axis is

     $$\binom{n}{r}\binom{n}{s} - \binom{n}{r-1}\binom{n}{s-1}.$$

     Solution: If the path x-axis goes below the x-axis, call $u = (x, -1)$ the first vertex where the path goes below the x-axis and reflect the path from $(0, 0)$ to u. This gives a path from $(0, -2)$ to (x, y). The number of such paths is $\binom{n}{r-1}\binom{n}{s-1}$.

   - Show that for the sequence

     $$\binom{n}{r}\binom{n}{0}, \binom{n}{r-1}\binom{n}{1}, \ldots, \binom{n}{0}\binom{n}{r}$$

     is unimodal. As $\binom{n}{r-j}\binom{n}{j} = \binom{n}{j}\binom{n}{r-j}$. We need to show that

     $$\binom{n}{r-j}\binom{n}{j} \leq \binom{n}{r-j-1}\binom{n}{j+1}$$

     for $j < r/2$. The number of paths from $(0, 0)$ to $(2j - r, n - r)$ is $\binom{n}{r-j}\binom{n}{j}$. Use the reflection principle.

2. Let a be the sequence $(a_0, \ldots, a_n)$. Here we suppose that $a_i = 0$ if $i < 0$ or $i > n$. The sequence a is a PF-sequence if the matrix $A = [a_{i-j}]$ is totally non negative, i.e. every square submatrix has a non negative determinant.

   - Prove that if a sequence is PF then it is log concave.
     Solution:

     $$\det \begin{pmatrix} a_i & a_{i+1} \\ a_{i-1} & a_i \end{pmatrix} = a_i^2 - a_{i-1}a_{i+1}$$

     .

   - Prove that the sequence $\binom{n}{0}, \ldots, \binom{n}{n}$ is PF. Use the Lindström-Gessel-Viennot lemma.
     Solution: Choose $A_i = (-i, i)$ and $B_j = (n - j, j)$. The number of paths from A_i to B_j is using N and E steps $a_{i,j} = \binom{n}{j-i}$. Therefore $\det(a_{ij})$ is the number of disjoint vertex path systems from $A_1, \ldots, A_n$ to $B_1, \ldots, B_n$. Such a system exists if the paths are from A_i to B_i, $i = 1, \ldots, n$. Hence $\det(a_{ij}) \geq 0$.

   - A sequence is a PF-sequence if and only if the polynomial $\sum_{k=0}^{n} a_k x^k$ is either constant or has only real zeros. Prove that $c(n, 0), \ldots, c(n, n)$ is a PF sequence. Here $c(n, k)$ is the number of permutations of $[n]$ into k cycles.

3. Prove that

   $$\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$$

   is a polynomial in q with non negative coefficients.

   Solution: Let $A_i = (-i, 0)$, $B_i = (k - i, n - k + i)$ for $i = 0, 1$ and weight the E steps from (x, y) to $(x + 1, y)$ by q^y.

Given $n \geq k$, let $A = (a_{i,j})_{1\leq i,j\leq n}$ be the matrix such that

$$a_{i,j} = \begin{bmatrix} n+j-1 \\ k+j-i \end{bmatrix}_q$$

Prove that every square submatrix of A has a determinant which is a polynomial in q with non negative coefficients.

Solution: Same for $i = 0, \ldots, n-1$.

4. Let $K_{m,n}$ be the complete bipartite graph with vertices $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ and edges (u_i, v_j) for all i, j. Prove that the number of spanning trees of $K_{m,n}$ is $m^{n-1}n^{m-1}$.

   Solution: Using the matrix tree theorem, the number of spanning trees is the determinant of a $(m+n-1) \times (m+n-1)$ matrix $M = (M_{i,j})$ with $M_{i,i} = n$ if $i \leq m$ and $M_{i,i} = m$ otherwise. And $M_{i,j} = -1$ if $i \leq m$ and $j > m$ or $i > m$ and $j \leq m$ and $M_{i,j} = 0$ otherwise. Using row operations, M can be transformed into an upper triangular matrix: add row $2, \ldots, m+n-1$ to row 1 and dd row 1 to row $m+1, \ldots, m+n-1$. The determinant is then $m^{n-1}n^{m-1}$.

5. A permutation $\pi \in \Sigma_n$ has inversion table $I(\pi) = (a_1, a_2, \ldots, a_n)$ where a_j is the number of elements of $Inv(\pi)$ of the form (i, j). Show that $0 \leq a_j < j$ for all j. Let

$$\mathcal{I}_n = \{a_1, a_2, \ldots, a_n) \mid 0 \leq a_j < j \text{ for all } j\}$$

   Show that the map $\pi \mapsto I(\pi)$ is a bijection $\Sigma_n \to \mathcal{I}_n$.

   Solution: The number of inversion sequences in $\mathcal{I}_n$ is $n!$. We just need to show that the map is surjective. Easy! $\pi_n = n - a_n$, i.e the $(a_n+1)st$ largest element of $[n]$ and continue setting π_i to be the $(a_i+1)st$ largest element of $[n] \backslash \{\pi_{i+1}, \ldots, \pi_n\}$ for i from $n-1$ downto 1.

6. Give a bijective and inductive proof of the following identity:

$$\prod_{i=0}^{n-1} \frac{1}{1-tq^i} = \sum_{k\geq 0} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k.$$

   Induction: true if $n = 0$.

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q = \begin{bmatrix} n+k-2 \\ k \end{bmatrix}_q + q^{n-1} \begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix}_q.$$

   Bijection: LHS is the generating function of partition into non negative parts less than n. The k^{th} term of the sum in the RHS is the generating function of partition into k non negative parts less than n.

7. Given $m \geq 2$, use generating functions to show that the number of partitions of n where each part is repeated fewer than m times equals the number of partitions of n into parts not divisible by m.

   Let $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$.

$$\prod_{i=1}^{\infty}(1+q^i+\ldots+q^{m-1}i) = \prod_{i=1}^{\infty}\frac{1-q^{im}}{1-q^i} = \frac{(q^m;q^m)_\infty}{(q;q)_\infty} = \prod_{i\geq 1,\ i\nmid m}\frac{1}{1-q^i}.$$

8. A directed animal A is a subset of $\mathbb{Z} \times \mathbb{N}$ such that
   - $(0,0) \in A$;
   - If $(x,y) \in A$ then $x \geq -y$;
   - If $(-i,i) \in A$ with $i > 0$ then $(-i+1, i-1) \in A$;
   - If $(x,y) \in A$ with $x > -y$ then $(x, y-1) \in A$ or $(x-1, y) \in A$.

The size of A is $|A|$. Prove that the number of directed animals A of size n is 3^{n-1}. Hint: Use a bijection.

Solution: change each brick by a vertex and rotate.

9. A brick configuration is a stack of 2×1 bricks such that

   - The bricks in the bottom row are contiguous
   - Every higher brick is supported by at least one brick in the row below it.

   Prove that the number of brick configurations with n bricks is 4^{n-1}. Hint: Use generating functions.

   Solution: Same as in class except $H(x) = x + 2xH(x) + xH(x)^2$.

10. Let $\mathcal{A}(G)$ be the set of acyclic orientations of the graph $G = (V, E)$. Given an edge $e \in E$. Prove that there is a bijection between $\mathcal{A}(G/e)$ where G/e is the graph where e is contracted to a vertex and the subset of the acyclic orientations in $\mathcal{A}(G\backslash e)$ such that when we add back e to this orientation, it can be oriented into both directions without creating any cycle.

11. Let G be a graph and t a positive integer. Call an acyclic orientation O and a not necessarily proper coloring $c : V \to [t]$ compatible if for each arc $(u, v) \in O$ we have $c(u) \leq c(v)$. Let $a(G, t)$ be the number of compatible pairs. Show if $|V| = n$ then

$$P(G; -t) = (-1)^n a(G, t).$$

Sketch: By induction on E. If $|E| = 0$ then true. As $a(G, t) = t^n$. Suppose $|E| > 0$. If O is such an orientation and $e = (u, v)$ an edge. If $c(u) \neq c(v)$ then we have one choice for the orientation. If $c(u) = c(v)$ then we could have one or two choices for the orientation. If we have two choices then this is counted by $a(G/e, t)$ (for one of the choice). The rest is counted by $a(G\backslash e, t)$. We get:

$$a(G, t) = a(G/E, t) + a(G\backslash e, t).$$

Using induction $P(G; -t) = (-1)^n a(G, t)$..