1. Consider lattice paths of length n, starting at the origin and ending at (x, y), and using steps N, E, S, W where $S = [0, -1]$ and $W = [-1, 0]$. Let $r = (n - x - y)/2$ and $s = (n + x - y)/2$.

   - Show that the number of such paths is given by
   $$\binom{n}{r}\binom{n}{s}$$
   - Show that the number of such paths staying weakly above the x-axis is
   $$\binom{n}{r}\binom{n}{s} - \binom{n}{r-1}\binom{n}{s-1}.$$
   - Show that for the sequence
   $$\binom{n}{r}\binom{n}{0}, \binom{n}{r-1}\binom{n}{1}, \ldots, \binom{n}{0}\binom{n}{r}$$
   is unimodal.

2. Let a be the sequence $(a_0, \ldots, a_n)$. Here we suppose that $a_i = 0$ if $i < 0$ or $i > n$. The sequence a is a PF-sequence if the matrix $A = [a_{i-j}]$ is totally non negative, i.e. every square submatrix has a non negative determinant.

   - Prove that if a sequence is PF then it is log concave
   - Prove that the sequence $\binom{n}{0}, \ldots, \binom{n}{n}$ is PF. Use the Lindström-Gessel-Viennot lemma.
   - A sequence is a PF-sequence if and only if the polynomial $\sum_{k=0}^{n} a_k x^k$ is either constant or has only real zeros. Prove that $c(n, 0), \ldots, c(n, n)$ is a PF sequence. Here $c(n, k)$ is the number of permutations of $[n]$ into k cycles.

3. Prove that
$$\left[\begin{array}{c} n \\ k \end{array}\right]_q \left[\begin{array}{c} n+1 \\ k \end{array}\right]_q - \left[\begin{array}{c} n \\ k-1 \end{array}\right]_q \left[\begin{array}{c} n+1 \\ k+1 \end{array}\right]_q$$
is a polynomial in q with non negative coefficients. Given $n \geq k$, let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be the matrix such that
$$a_{i,j} = \left[\begin{array}{c} n+j-1 \\ k+j-i \end{array}\right]_q$$
Prove that every square submatrix of A has a determinant which is a polynomial in q with non negative coefficients.

4. Let $K_{m,n}$ be the complete bipartite graph with vertices $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ and edges (u_i, v_j) for all i, j. Prove that the number of spanning trees of $K_{m,n}$ is $m^{n-1}n^{m-1}$.

5. A permutation $\pi \in \Sigma_n$ has inversion table $I(\pi) = (a_1, a_2, \ldots, a_n)$ where a_j is the number of elements of $Inv(\pi)$ of the form (i, j). Show that $0 \leq a_j < j$ for all j. Let
$$\mathcal{I}_n = \{a_1, a_2, \ldots, a_n) \mid 0 \leq a_j < j \text{ for all } j\}$$
Show that the map $\pi \mapsto I(\pi)$ is a bijection $\Sigma_n \to \mathcal{I}_n$.

6. Give a bijective and inductive proof of the following identity:
$$\prod_{i=0}^{n-1} \frac{1}{1 - tq^i} = \sum_{k \geq 0} \left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_q t^k.$$

7. Given $m \geq 2$, use generating functions to show that the number of partitions of n where each part is repeated fewer than m times equals the number of partitions of n into parts not divisible by m.

8. A directed animal A is a subset of $\mathbb{Z} \times \mathbb{N}$ such that

   - $(0, 0) \in A$;
   - If $(x, y) \in A$ then $x \geq -y$;
   - If $(-i, i) \in A$ with $i > 0$ then $(-i + 1, i - 1) \in A$;
   - If $(x, y) \in A$ with $x > -y$ then $(x, y - 1) \in A$ or $(x - 1, y) \in A$.

   The size of A is $|A|$. Prove that the number of directed animals A of size n is 3^{n-1}. Hint: Use a bijection.

9. A brick configuration is a stack of 2×1 bricks such that

   - The bricks in the bottom row are contiguous
   - Every higher brick is supported by at least one brick in the row below it.

   Prove that the number of brick configurations with n bricks is 4^{n-1}. Hint: Use generating functions.

10. Let $\mathcal{A}(G)$ be the set of acyclic orientations of the graph $G = (V, E)$. Given an edge $e \in E$. Prove that there is a bijection between $\mathcal{A}(G/e)$ where G/e is the graph where e is contracted to a vertex and the subset of the acyclic orientations in $\mathcal{A}(G \backslash e)$ such that when we add back e to this orientation, it can be oriented into both directions without creating any cycle.

11. Let G be a graph and t a positive integer. Call an acyclic orientation O and a not necessarily proper coloring $c : V \to [t]$ compatible if for each arc $(u, v) \in O$ we have $c(u) \leq c(v)$. Let $a(G, t)$ be the number of compatible pairs. Show if $|V| = n$ then

$$P(G; -t) = (-1)^n a(G, t).$$