1. Let $\mathcal{C}_n$ be the number of tilings of n boxes arranged in a circle with dominoes and monominoes and let $L_n = |\mathcal{C}_n|$.
   (a) Prove that $L_n = F_{n-1} + F_{n+1}$ for $n \geq 1$.
   (b) Prove that $L_{m+n} = F_{m-1}L_n + F_mL_{n+1}$.
   (c) Prove that $F_{2n} = F_nL_n$.

2. A permutation of a multiset $M = \{\{1_1^n, \dots, m^{n_m}\}\}$ is a linear arrangement of the elements of M. For example, the permutation of $\{\{1, 2, 2\}\}$ are 122, 212 and 221. Prove in three different ways that if $n_1 + \ldots + n_m = n$ then the number of permutations of M is
$$\binom{n}{n_1, \dots, n_m} = \frac{n!}{n_1! \dots n_m!}.$$
   (For example: induction on n, induction on m, combinatorial argument...)

3. Prove that the Prüfer algorithm is a bijection.

4. Let $\alpha = (a_1, \dots a_n)$ be a sequence with $a_i \in [n]$, $1 \leq i \leq n$. Suppose a street has n parking spaces labelled $1, 2, \dots, n$. For $i = 1, 2, \dots, n$, car i tries to park in space a_i and if it is occupied then it takes the next available space. We say that α is a parking function of length n is all the cars can park.
   (a) Let $b_1 \leq b_2 \leq \ldots \leq b_n$ be the increasing rearrangement of α. Prove that α is a parking function if and only $b_i \leq i$.
   (b) Prove that the number of parking functions of length n is $(n+1)^{n-1}$. Hint: Add an additional space $n+1$, and arrange the spaces in a circle. Allow $n+1$ also as a preferred space. Now all cars can park, and there will be one empty space. α is a parking function if and only if the empty space is $n+1$.

5. Show that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ also count:
   (a) Words $w = w_1w_2 \dots w_{2n}$ containing n ones and n zeroes such that in any prefix $w_1 \dots w_i$ the number of ones is greater or equal to the number of zeroes.
   (b) Sequences of positive integers $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n$ with $a_i \leq i$.

6. A derangement is a permutation with no fixed points (or equivalently no cycles of length one). Let $D(n)$ be the number of derangements of $[n]$. Prove that for $n \geq 2$
$$D(n) = (n-1)(D(n-1) + D(n-2));$$
   and that $D(1) = 0$ and $D(0) = 1$. Prove this in two ways using the principle of inclusion-exclusion and a combinatorial argument.

7. In how many ways can n couples sit around a table so that no one sits next to its partner? Use the principle of inclusion-exclusion.

8. Let $c(n, k)$ be the number of permutations of $[n]$ into k cycles. Prove that $c(1, 1) = 1$ and that for $n \geq 2$
$$\sum_{k=1}^{n} (-1)^{n-k} c(n, k) = 0$$
   in two ways: by induction and by using a sign reversing involution.

9. Let $P_d(n)$ (resp. $P_o(n)$) be the set of partitions of n into distinct (resp. odd) parts. Let $g : P_d(n) \to P_o(n)$ be the map such that for any $\lambda \in P_d(n)$ we write each part $p = q2^r$ with q odd and $r \geq 0$. We replace p by 2^r copies of q and obtain a partition into $P_o(n)$. Prove that g is a bijection and that it is identical to the bijection obtained by the Involution Principle (optional). Generalize this to prove that given $m \geq 2$, the number of partitions of n where each part is repeated less than m times is equal to the number of partitions of n where none of the parts are divisible by m.