Math 142

Homework 9 – Due April 24, 2018

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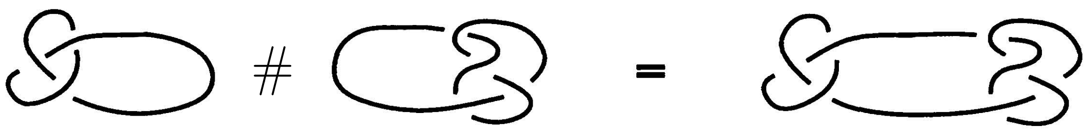
1. Armstrong, page 223 #12 (it may help to do some of the below problems first)
2. There is an operation on knots called the *connected sum*. It is defined on Armstrong, page 225 (first paragraph; ignore the first sentence). Although in the book it is written as $K + K'$, it is more common nowadays to see it written as $K\#K'$. It's not proved there, but this operation is well-defined, commutative, and associative on isotopy classes of knots.[1] The below image captures what's happening.



   (a) Let $c_3(K)$ be the number of 3–colourings of K (including the trivial colourings). Show that $c_3(K)$ is a power of three.

   (b) Find a formula for $c_3(K\#K')$ in terms of $c_3(K)$ and $c_3(K')$.

   (c) If K is a 3–colourable knot, show that $K\#K$ is not isotopic to K. (This might help with Armstrong page 223 #12.)

3. The *crossing number* $c(K)$ of a knot or link is the minimal number of crossings that can appear in a nice diagram of K.

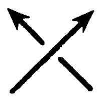
   (a) What is the crossing number of the unknot? Show that if K is a knot, then $c(K) \leq 2$ implies that K is the unknot ("is" in this context means "is isotopic to"). Show that $c(K) = 3$ if and only if K is (one of the two) trefoils. If you're brave, show that $c(K) = 4$ if and only if K is the figure-eight knot.

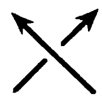
   Hint: draw the crossings first, and then figure out how to connect the loose ends to get a knot. Remember that for this question, you're looking for a knot, not a link, and once you've drawn the crossings, no other edges can cross.

   (b) Show that $c(K\#K') \leq c(K) + c(K')$ (assuming the connected sum operation is well defined). (Showing that equality holds is an open problem.)

[1] Think about it! To define this more generally than in the textbook, you want to take the connected sum of two copies of S^3, one containing K, the other containing K'. Choose the balls such that they intersect the knots in a simple arc (homeomorphic to the vertical line between antipidal points). Then choose the homeomorphism of the boundary spheres such that the points where the arcs hit the spheres are identified appropriately (*ie.* there are two choices, and you want to choose the one that doesn't give you a twist.

4. Given a link with two components (*ie.* a subset of $\mathbb{R}^3$ homeomorphic to $S^1 \coprod S^1$), call the components K_1 and K_2. Choose a direction for each component (it's now called an *oriented link*). For every crossing that involves K_1 and K_2, we give a value ± 1 (see the picture – you may need to rotate your diagram in order to compare some of your crossings to the picture). We ignore all the crossings that only involve one component. Define the *linking number* $lk(K_1, K_2) = lk(K_2, K_1)$ to be half of the sum of the values over all the crossings.

+1 

 **- 1**

   (a) Show that $lk(K_1, K_2)$ is an invariant of oriented links. That is, show that it's invariant under Reidemeister moves.

   (b) Show that if we change the orientation of exactly one of the components, then the linking number changes sign.

   (c) Show that there are exactly three 2–component oriented links up to isotopy that have diagrams with two or fewer crossings, and that they're distinguished by their linking numbers.