Math 142

Homework 10 – Not graded

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1. Armstrong, page 222 #1, #2 (see Figure 10.6 and the second paragraph of p222).

2. Show that if K is n–colourable, then K is mn–colourable for any $m \in \mathbb{N}$.

3. Given a knot K and a nice projection P of K with n arcs, make an $n \times n$ matrix M_P representing the n–colouring relations ($2\ell_c - \ell_a - \ell_b \equiv 0 \mod n$ at every crossing). For example, the trefoil with its standard picture might have the matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

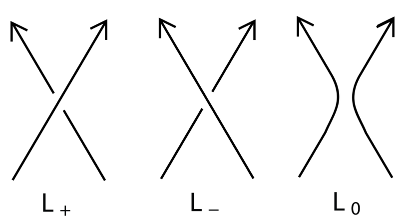
   (a) Show that $\det M_P = 0$.

   (b) Let $\lambda_1, \ldots, \lambda_m$ be the non-zero eigenvalues of M_P, and let $d(K) = |\lambda_1 \cdots \lambda_m|$. For a prime number p, show that K is p–colourable if and only if p divides $d(K)$. (This value $d(K)$ is called the *determinant* of K, and is independent of the projection.)

4. We want to distinguish the two trefoils. We will define an invariant $V_P(t)$ of projections P that doesn't depend on the projection P (although showing that is hard!). Given an oriented projection P of a link, look at a particular crossing. We can form three different projections P_+, P_-, and P_0 (of potentially different links!) by replacing the crossing with one of the three local pictures below (note that P is equal to either P_+ or P_-). We can define a polynomial $V_P(t)$ by setting

$$t^{-1}V_{P_+}(t) - tV_{P_-}(t) = (t^{1/2} - t^{-1/2})V_{P_0}(t).$$

   We also want to define $V_P(t) = 1$ for any projection P of the unknot.



   (a) Calculate $V_P(t) = -(t^{1/2} + t^{-1/2})$, where P is the projection of two unlinked unknots with no crossings. *Hint: start with a projection of the unknot with a single crossing.* Use this to calculate $V_P(t)$ for the projections of the 2–component links with two crossings.

   (b) Calculate $V_P(t)$ for the usual projections of the two trefoils, and notice that they're not the same.

5. This is from Rolfsen's *Knots and Links*, on page 66. Look at "5. A MAGIC TRICK".

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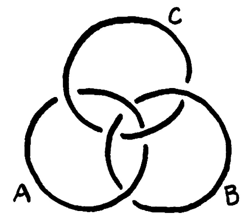
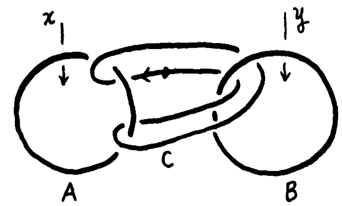
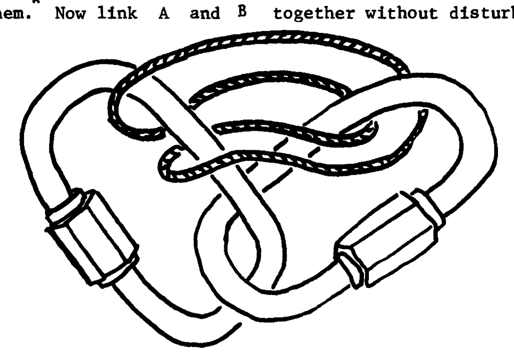
4. EXAMPLE : The <u>Borromean rings</u> . Note that any two components form a trivial link. Nevertheless, it is not trivial. In fact any component, say C , is a homotopically nontrivial loop in the complement of the other two. For the link is equivalent to this link : And C represents a commutator $xyx^{-1}y^{-1}$ of the two generators of $\pi_1(R^3) - (A \cup B))$. Since this group is not abelian, $C \not\simeq *$.





5. A MAGIC TRICK. Take two rings which can be opened up (mountaineering caribiners would be fine) and arrange them as A and B above. Link a piece of string about them in the manner of C, and ask your audience to undo them.* Now link A and B together without disturbing C .



Then C can be slipped off A and B ! Explain.

* without opening the rings, of course.