MATH 215A
Homework 8 – Not due
Jamie Conway

(1) Look at the following problems from Hatcher:
- page 204: # 6, 8, 9
- page 228–9: # 1, 4, 9
- page 258: # 20, 24, 25, 26

(2) A cochain $\phi \in C^1(X;G)$ can be thought of as a function from paths in X to the group G (and is in fact such a function extended linearly over the entirety of $C_1(X)$). Show that if $d'\phi = 0$, then the following properties hold.

(a) $\phi(\gamma \cdot \gamma') = \phi(\gamma) + \phi(\gamma')$.

(b) If γ is homotopic to γ', then $\phi(\gamma) = \phi(\gamma')$.

(c) ϕ is a coboundary if and only if $\phi(\gamma)$ only depends on the endpoints of γ, for all γ.

Show that the above properties allow you to write a homomorphism $H^1(X;G) \to \mathrm{Hom}(\pi_1(X), G)$. (Note that homomorphisms from $\pi_1(X)$ to Abelian groups factor through $\mathrm{Ab}(\pi_1(X))$, so if X is path-connected, then the universal coefficients theorem says that this is an isomorphism.)

(3) It can be shown that $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[a]/\langle a^{n+1}\rangle$, where a is a generator of $H^2(\mathbb{CP}^n)$. Show that this implies that there is no orientation-reversing homeomorphism $\mathbb{CP}^2 \to \mathbb{CP}^2$. How about $\mathbb{CP}^n \to \mathbb{CP}^n$ for other n?

(4) It can be shown that $H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[a]/\langle a^{n+1}\rangle$, where a is a generator of $H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$. Show that there is no map $\mathbb{RP}^n \to \mathbb{RP}^m$ inducing a non-trivial map $H^1(\mathbb{RP}^m;\mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$ if $n > m$.

(5) Using cup products, show that any map $S^{k+l} \to S^k \times S^l$ induces the trivial map $H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$, when $k, l > 0$. Show that this is not necessarily true for maps $S^k \times S^l \to S^{k+l}$.

(6) Compute all cap products in $S^n \times S^m$ (don't forget the case of $n = m$).

(7) (a) The *degree* of a map $f : M \to N$ between closed oriented n–manifolds with orientation classes $[M] \in H_n(M)$ and $[N] \in H_n(N)$ is the value $d \in \mathbb{Z}$ such that $f_*([M]) = d[N]$. Show that there is a degree–1 map $M \to S^n$ for any connected, closed, orientable n–manifold M.

(b) If A is a 2×2 matrix with integer entries, then A induces a map $f_A : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, that is, from T^2 to itself. Show that under the obvious identification $H_1(T^2) \cong \mathbb{Z}^2$ (paths from the origin along the coordinate axes), f_A^* is the transpose of A.

(c) Show that the degree of f_A is equal to $\det A$. (Use the cup product structure on T^2 and naturality.)

(8) (a) Show that any simply-connected, closed n–manifold is orientable (just show that $H_n(M) \cong \mathbb{Z}$). (Assume not, then use $\mathbb{Z}/2\mathbb{Z}$ Poincaré Duality and the UCT.)

(b) Show that if M is a closed, simply-connected (hence oriented) 4–manifold, then $H_2(M)$ has no torsion.