MATH 215A
Homework 7 – Due November 15, 2018
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Work on all of these problems, but carefully write up and turn in only problems 2, 4, 5, 6.

Feel free (and encouraged!) to work with your classmates on this homework, but you **must** write up your own solutions. Indicate on your homework the set of people with whom you worked, if that set is non-empty, and cite any other sources you consulted besides your notes and your textbook.

(1) If X is a CW complex that is the union of Y and Z, where Y, Z, and $Y \cap Z$ are subcomplexes, then show that the Euler characteristic satisfies $\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z)$.

(2) If $\widetilde{X}$ is a degree d covering space of a CW complex X, then show that $\chi(\widetilde{X}) = d\chi(X)$. Show that a closed surface of genus g covers a surface of genus h if and only if $g = n(h-1)+1$ for some positive integer n.

(3) We say that a graph Γ (that has no multiple edges or loops) is *cellularly embedded* in a closed surface S if there is a CW structure for S where the 1–skeleton is isomorphic to Γ. Given such a CW structure, let V, E, and F denote the number of 0–, 1–, and 2–cells, respectively. Equivalently, this is an embedding $\phi : G \to S$ such that $S \backslash \phi(G)$ is the disjoint union of open discs.

(a) By summing up the number of edges in each face, show that $2E \geq 3F$ (assume the graph has at least two edges — the other cases are trivial for what we want to do below). By summing up the number of edges at each vertex, show that that $2E \geq \delta$, where δ is the minimal *degree* of a vertex G (the degree of a vertex is the number of edges ending at the vertex).

(b) Show that if G is cellularly embedded in S, then $V - E + F = \chi(S)$.

(c) Let K_n be the graph on n vertex where each vertex is connected to every other by an edge (the *complete graph on n vertices*). Show that there a cellular embedding of K_1, $\ldots$, K_4 in S^2, but that K_5 does not cellularly embed in S^2. We say that K_5 is not *planar.*

(Look up the bipartite graph $K_{3,3}$, and show that it also does not cellularly embed in S^2. You'll have to get a better inequality in part (a).)

(d) Show that K_5, K_6, and K_7 cellularly embed in T^2, but that K_8 does not.

(4) Find an example of spaces X and Y and maps $f, g : X \to Y$ such that f and g induce the same map $H_n(X) \to H_n(Y)$ for every n, but induce different maps $H_n(X;G) \to H_n(Y;G)$ for some n and some G. Why doesn't this contradict the universal coefficient theorem? What's going on here?

(5) If G is a finitely generated Abelian group, show that $\mathrm{Tor}(G, \mathbb{Q}/\mathbb{Z})$ is isomorphic to the torsion subgroup of G. Hence show that G is torsion-free if and only if $Tor(G, G') = 0$ for all G'.

(6) If $0 \to G_1 \to G_2 \to G_3 \to 0$ is a short exact sequence of Abelian groups, show that for any space X there is a long exact sequence

$$\cdots \to H_{n+1}(X;G_3) \to H_n(X;G_1) \to H_n(X;G_2) \to H_n(X;G_3) \to H_{n-1}(X;G_1) \to \cdots$$

The boundary map in this long exact sequence is called the *Bockstein homomorphism.*