MATH 215A

Homework 6 – Due November 6, 2018

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*Work on all of these problems, but carefully write up and turn in only problems 2, 5, 6, 7*

Feel free (and encouraged!) to work with your classmates on this homework, but you **must** write up your own solutions. Indicate on your homework the set of people with whom you worked, if that set is non-empty, and cite any other sources you consulted besides your notes and your textbook.

(1) Do the following problems from Hatcher:

- page 132: # 16, 17, 28, 29
- page 155-6: # 2, 9

(2) We know that if $p : \widetilde{X} \to X$ is a covering space map, then the induced map $p_*$ on $\pi_1$ is injective. Show that the induced map on $H_1$ need not be injective. *Hint:* consider an example with $S^1 \vee S^1$.

(3) Let $(A_*, d_A)$ and $(B_*, d_B)$ be two chain complexes and let $f : A_* \to B_*$ be a chain map. Define a new complex $(M_*(f), d_f)$ called the *mapping cone* of $f$, where $M_n(f) = A_{n-1} \oplus B_n$ and

$$d_f(a, b) = (-d_A a, d_B b + f_{n-1}(a)) \text{ for } (a, b) \in M_n(f).$$

Show that this defines a chain complex and that there is an exact sequence

$$\cdots \to H_n(A_*, d_A) \to H_n(B_*, d_B) \to H_n(M_*(f), d_f) \to H_{n-1}(A_*, d_A) \to \cdots .$$

Moreover, show that the arrow $H_n(A_*, d_A) \to H_n(B_*, d_B)$ in the long exact sequence is induced by $f$, and hence that $f$ induces an isomorphism on homology if and only if $H_n(M_*(f), d_f) = 0$ for all $n$.

*Hint:* there is a natural inclusion $B_n \to M_n(f)$ that is a chain map. If we let $A_*^-$ be the complex with $A_n^- = A_{n-1}$ and $d_{A^-}a = -d_A a$, then there is a natural projection map $M_n(f) \to A_*^-$. Note that $H_n(A_*^-) \cong H_{n-1}(A_*)$.

(4) Consider the simplicial chain complex $S_*(\Delta^n)$ of the $n$–simplex $\Delta^n$.

(a) What is the rank of the group $S_k(\Delta^n)$?

(b) Let $X^k$ be the $k$–skeleton of $\Delta^n$ (that is, the union of all the $i$–dimensional faces of $\Delta^n$ for $i \leq k$). Compute $\widetilde{H}_*(X^k)$ (without using cellular homology).

(5) Given a space $X$, define the *suspension* of $X$ to be $SX = (X \times I)/\sim$, where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$ ($SX$ is homeomorphic to $X * S^0$). Given a map $f : X \to Y$, we get an induced map $Sf : SX \to SY$ by $[(x, t)] \mapsto [(f(x), t)]$.

(a) Prove that $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ for all $n$. Prove this twice: once with Mayer–Vietoris and once without (but for your homework, only turn in the one that doesn't use M–V).

(b) If $\phi$ is the isomorphism from (a), and $f : X \to Y$ is a continuous function, then show that the following diagram commutes.

$$\begin{array}{ccc} \widetilde{H}_n(X) & \xrightarrow{f_*} & \widetilde{H}_n(Y) \\ \downarrow \phi & & \downarrow \phi \\ \widetilde{H}_{n+1}(SX) & \xrightarrow{(Sf)_*} & \widetilde{H}_{n+1}(SY). \end{array}$$

(c) Show that for each $n$ and each $d \in \mathbb{Z}$, there exists a map $f : S^n \to S^n$ with $\deg f = d$.

(6) (a) Compute $H_*(\mathbb{RP}^n/\mathbb{RP}^m)$ for $m < n$ by cellular homology, using the standard CW complex structure on $\mathbb{RP}^n$ with $\mathbb{RP}^m$ as its $m$–skeleton.

(b) If $X$ is a finite cell complex of dimension $n$, and $m > n$, then what is the homology of $X \times S^m$? You might want to look back at HW1 Q6.

(7) Suppose $K \subset S^3$ is a *tame knot*, *ie.* $K$ is a subspace of $S^3$ homeomorphic to $S^1$ that has a neighbourhood $N$ homeomorphic to $S^1 \times D^2$ such that $K = S^1 \times \{0\}$. Use the Mayer–Vietoris sequence to compute $H_*(S^3 - \operatorname{int} N)$.

(8) Following on from the previous question, let $M_K$ be the space $S^3 - \operatorname{int} N$. Similarly to HW4 Q4, put coordinates on $\partial M_K$ such that $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ corresponds to $\{\mathrm{pt}\} \times \partial D^2$ and $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ corresponds to some curve on the torus that is isotopic in $N$ to $K$. Now, form a new space $M_K \cup_f S^1 \times D^2$ by gluing a solid torus to $M_K$ via a homeomorphism $f : \partial\left(S^1 \times D^2\right) \to \partial M_K$, as in HW4 Q4.

Use Mayer–Vietoris to calculate the homology of the resulting space. Hence show that you can create *homology spheres* this way, *ie.* 3–manifolds with the same homology as $S^3$. (You'd need to calculate $\pi_1$ — or some other invariant — to show that you're not always getting $S^3$.)

(9) Let $S$ be a closed surface. An *isotopy* of homeomorphisms is a homotopy $h_t$ that is a homeomorphism at each $t$. Let $\operatorname{Homeo}_0(S)$ be the subgroup of homeomorphisms of $S$ isotopic to the identity. Define the *mapping class group* of $S$ to be

$$\operatorname{Mod}(S) = \operatorname{Homeo}(S)/\operatorname{Homeo}_0(S).$$

Now let $S$ be the torus $T^2$, and fix an identification of $H_1(T^2)$ with $\mathbb{Z}^2$. Show that the homomorphism

$$\sigma : \operatorname{Mod}(T^2) \to GL(2, \mathbb{Z})$$

given by the action on $H_1(T^2)$ is injective and has image equal to $SL(2, \mathbb{Z})$. *Hint: for injectivity, use the fact that $T^2$ is a $K(\mathbb{Z}^2, 1)$.*