MATH 215A

Homework 5 – Due October 11, 2018

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Work on all of these problems, but carefully write up and turn in only problems 1bc, 2, 3, 4

Feel free (and encouraged!) to work with your classmates on this homework, but you **must** write up your own solutions. Indicate on your homework the set of people with whom you worked, if that set is non-empty, and cite any other sources you consulted besides your notes and your textbook.

(1) A chain complex C_* is *exact* at C_n if $H_n(C) = 0$. The chain complex is *exact* if $H_n(C) = 0$ for all n.

(a) Given a sequence $A \to B \to C \to D \to E$ that is exact at B, C, and D, show that $C = 0$ if and only if the map $A \to B$ is surjective and the map $D \to E$ is injective.

(b) Show that if $0 \to A \to B \to \mathbb{Z}^k \to 0$ is exact, then $B \cong A \oplus Z^k$.

(c) Determine whether there exists an exact chain complex

$$0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to 0.$$

More generally, determine which Abelian groups A fit into an exact chain complex

$$0 \to \mathbb{Z}/p^m\mathbb{Z} \to A \to \mathbb{Z}/p^n\mathbb{Z} \to 0,$$

with p prime. What about the case of an exact chain complex

$$0 \to \mathbb{Z} \to A \to \mathbb{Z}/n\mathbb{Z} \to 0?$$

(2) Let R be a commutative ring with 1, and let M be an R–module. Construct a chain complex (C_*, d) over R such that $C_k \equiv 0$ for all $k < 0$, C_k is a free R–module for all $k \geq 0$, and

$$H_*(C) = \begin{cases} M & * = 0, \\ 0 & * \neq 0. \end{cases}$$

(3) Let X be a topological space.

(a) Prove that constant paths are null-homologous. That is, $[e_\gamma] = 0 \in H_1(X)$ for any constant path $\gamma : I \to X$.

(b) If γ_1 and γ_2 are paths such that $\gamma_1(1) = \gamma_2(0)$, prove that $[e_{\gamma_1 \cdot \gamma_2} - e_{\gamma_2} - e_{\gamma_1}] = 0$.

(c) If γ is a path and $\overline{\gamma}(t) = \gamma(1-t)$, prove that $[e_{\overline{\gamma}}] = -[e_\gamma]$.

(4) For each natural number n, let $\tau_n : \Delta^n \to \Delta^n$ be the map $(t_0, \ldots, t_n) \mapsto (t_n, \ldots, t_0)$. Given a space X, define a map $f_n : C_n(X) \to C_n(X)$ for each n by

$$f_n(e_\sigma) = (-1)^{n(n+1)/2} e_{\sigma \circ \tau_n}.$$

Prove that $f = \oplus f_n$ defines a chain map $C_*(X) \to C_*(X)$.

(5) Suppose X is a finite simplicial complex (that is, a Δ–complex where each simplex is uniquely determined by its vertices), and let CX be the cone on X. Let v_0 be the vertex of the cone (that is, the vertex that is not in the image of the natural inclusion $X \subset CX$). Show that the chain map $\iota_\# : C_*(v_0) \to C_*(CX)$ induced by the inclusion of the vertex v_0 is a chain homotopy equivalence (that is, there exists a chain map $r_\# : C_*(CX) \to C_*(v_0)$ such that their appropriate compositions are chain homotopic to the identity maps). You should not use the homotopy invariance of homology for this.