MATH 215A
Homework 4 – Due October 4, 2018
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Work on all of these problems, but carefully write up and turn in only problems 2, 4, 6, 9bcd.

Feel free (and encouraged!) to work with your classmates on this homework, but you **must** write up your own solutions. Indicate on your homework the set of people with whom you worked, if that set is non-empty, and cite any other sources you consulted besides your notes and your textbook.

(1) Read Chapter 1.2 of Hatcher, and do the following problems:
- page 52: # 3,7,8,9,16,17,22

(2) Consider a connected CW complex X with one 0–cell x_0. Explain (using words and mathematical notation) how to give a presentation for $\pi(X, x_0)$. You should explain how the 1–cells, 2–cells, 3–cells, ..., play a role, and why.

(3) The lens space $L(p, q)$, for $p > 1$ and p and q relatively prime, is the quotient space $S^3/(\mathbb{Z}/p\mathbb{Z})$, where S^3 is the unit sphere in $\mathbb{C}^2$, and a generator g of $\mathbb{Z}/p\mathbb{Z}$ acts on (z_1, z_2) as

$$g(z_1, z_2) = (e^{2\pi i/p} z_1, e^{2\pi iq/p} z_2).$$

(a) Prove that $L(p, q) \not\simeq L(p', q')$ for $p \neq p'$.

(b) Prove that $L(p, q) \cong L(p, q')$ when $q \equiv \pm q' \bmod p$ or $qq' \equiv \pm 1 \bmod p$.

It turns out that the condition in (b) is an "if and only if", but it's not easy to prove.

(4) Let A and B be two solid tori (homeomorphic to $S^1 \times D^2$), and fix an identification

$$\partial A \cong \partial B \cong T^2 = \mathbb{R}^2/\mathbb{Z}^2,$$

such that the straight line in $\mathbb{R}^2$ from $(0, 0)$ to $(0, 1)$ corresponds to a curve in T^2 that bounds a disc in A and in B, and that loop along with the image of the straight line from $(0, 0)$ to $(1, 0)$ generate $\pi_1(T^2, (0, 0))$.

Construct a space M_f (in fact, a 3–manifold) by gluing ∂A to ∂B via a map (defined on $\mathbb{R}^2$ that descends to T^2)

$$f = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \text{ such that } ps - qr = 1.$$

(a) Construct S^3 as such a manifold. That is, find such a map f such that M_f is homeomorphic to S^3. (Hint: consider HW1 Q3) Do the same for $S^1 \times S^2$.

(b) Calculate $\pi_1(M_f)$. Your result should only depend on one of the variables in the matrix.

(5) *(We didn't get to this material, but if you want to think about it, read pages 91ff)* Consider a graph of groups where all the vertices are labeled with the trivial group. Show that the fundamental group of the graph of groups is free.

(6) Given a map $f : X \to X$, the *mapping torus* T_f of f is the space obtained from $X \times [0, 1]$ by identifying $(x, 1)$ with $(f(x), 0)$ for all $x \in X$. If $X = S^1 \vee S^1$, and f is a basepoint-preserving map, write a presentation for $\pi_1(T_f)$ in terms of the map $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$. Do the same for $X = S^1 \times S^1$.

(7) Let X be simply-connected and locally path-connected, and let G_1 and G_2 be two subgroups of $\text{Homeo}(X)$ that give covering space actions on X. Show that the orbit spaces X/G_1 and X/G_2 are homeomorphic if and only if G_1 and G_2 are conjugate subgroups of $\text{Homeo}(X)$.

(8) Let X and Y be two non-empty spaces, and assume that X is path-connected. Show that the join $X * Y$ is simply connected.

(9) Let K be the Klein bottle. Recall the K is constructed from the cylinder $S^1 \times [0, 1]$ by identifying the point $(x, 0)$ with $(r(x), 1)$, where $r : S^1 \to S^1$ is some reflection of the circle.

  (a) Construct a covering map $p : T^2 \to K$. Describe the group of deck transformations.

  (b) Construct a covering map $p : \mathbb{R}^2 \to K$. Describe the group of deck transformations.

  (c) Using (b), show that $\pi_1(K)$ is isomorphic to the group G whose elements are pairs $(a, b) \in \mathbb{Z}^2$ with group operation
$$(a, b) \cdot (c, d) = (a + (-1)^b c, b + d).$$

  (d) Describe K as $(S^1 \vee S^1) \cup_f D^2$ for some map $f : S^1 \to S^1 \vee S^1$. Use this description to give a presentation for $\pi_1(K)$ with two generators and one relation. Show directly that this presentation is isomorphic to G.