MATH 215A

Homework 3 – Due September 20, 2018

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Work on all of these problems, but carefully write up and turn in only problems 2, 4, 5, 7.

Feel free (and encouraged!) to work with your classmates on this homework, but you **must** write up your own solutions. Indicate on your homework the set of people with whom you worked, if that set is non-empty, and cite any other sources you consulted besides your notes and your textbook.

Note: For this homework, you may assume that the fundamental group of the wedge of n circles based at their common point x of intersection is the free group F_n on n generators, where a generating set is $[\gamma_i]$, where γ_i traverses the i^{th} circle once. You may, if desired, use the Seifert–van Kampen theorem (Theorem 1.20 in Hatcher) without proof.

(1) Read Chapter 1.3 of Hatcher, and do the following problems:

- page 79: # 5, 10, 11, 12, 13, 14, 18, 19
  (you may want to look at Example 1.21 for the fundamental group of wedge products)

(2) Let X be a path-connected, locally path-connected space, and let $\pi_1(X)$ be finite. Show that every continuous map $X \to S^1$ is null-homotopic.

(3) Let X be a path-connected, locally path-connected, semi-locally simply connected space (so: X has a universal cover). Prove that path-connected covering spaces of X of degree n correspond to representations of $\pi_1(X)$ to the symmetric group S_n that act transitively on $\{1, \ldots, n\}$. In particular, given a path-connected cover $\widetilde{X} \to X$, show that there is an associated homomorphism $h : \pi_1(X) \to S_n$ such that for each i, j, there is some $g \in \pi_1(X)$ such that $h(g)(i) = j$, and conversely, given such an h, there is a path-connected covering space realising this homomorphism.

(4) Using the previous problem, one can describe a degree–n covering space of $S^1 \vee S^1$ by labeling the loops by elements of S_n, such that the two elements generate a transitive subgroup of S_n. Draw the covering spaces for: (a) (1 2) and id, in S_2; (b) (1 2 3) and id, in S_3; (b) (1 2) and (2 3), in S_3.

(5) Let $p : X \to Y$ and $q : Y \to Z$ be covering space maps of finite degree. Show that $q \circ p : X \to Z$ is a covering space map.

(6) A *graph* G is a 1–dimensional CW–complex. A *tree* is a contractible graph. A tree T inside a graph G is *maximal* if no strictly larger subgraph is a tree. You may assume that every graph has a maximal tree.

   (a) If $T \subset G$ is a tree, show that $G \to G/T$ is a homotopy equivalence. Hence, show that every connected graph is homotopy equivalent to a graph with a single vertex.

   (b) Prove that every covering space of a graph is also a graph.

   (c) Use (a) and (b) to prove that every subgroup of a free group is a free group.

(7) Show that a finitely generated group has a finite number of subgroups of index n, for a fixed n. *Hint: consider first the case of free groups using graphs and covering space theory, using the previous question; then prove the general result, using the fact that any finitely generated group is the quotient of a free group.*