MATH 215A

Homework 2 – Due September 13, 2018

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Work on all of these problems, but carefully write up and turn in only problems 2, 3, 5, 6.

Feel free (and encouraged!) to work with your classmates on this homework, but you **must** write up your own solutions. Indicate on your homework the set of people with whom you worked, if that set is non-empty, and cite any other sources you consulted besides your notes and your textbook.

(1) Read Chapter 1.1, 1.3 of Hatcher, and do the following problems:
- pages 38–40: # 3, 6, 7, 8, 11, 12, 16, 17, 20
- page 79: # 1, 2

(2) If X is a space with a contractible universal cover, then show that any map $S^n \to X$, for $n \geq 2$, can be extended to a map $D^{n+1} \to X$.

(3) Construct a simply-connected covering space of the subspace X of $\mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle that intersects it at exactly two points. A picture is sufficient here, with a short description of the projection map.

(4) Construct a covering map $p : \mathbb{R}^2 \to K$ of the Klein bottle, and hence show that $\pi_1 K$ is isomorphic to the group G with elements $(m, n) \in \mathbb{Z}^2$ and group operation

$$(m, n) \cdot (p, q) = (m + (-1)^n p, n + q).$$

Show that K has a covering space homeomorphic to the torus T^2, but that T^2 does not have a covering space homeomorphic to K.

(5) Let X be a space, let $p \in X$, and let $\Sigma = T^2 - \text{interior}(\mathrm{D}^2)$ (that is, the complement of the interior of some embedded disc in the torus). Consider a continuous map $\gamma : \partial\Sigma \to X$ with $\gamma(1) = p$ (where we think of $S^1 = \partial\Sigma$ as being the unit sphere in $\mathbb{C}$). Note that γ can be thought of as a path $t \mapsto \gamma(e^{2\pi i t})$.

(a) Suppose that γ extends to a map $\Gamma : \Sigma \to X$ (*ie.* $\Gamma|_{\partial\Sigma} = \gamma$). Show that $[\gamma] = aba^{-1}b^{-1}$, for some $a, b \in \pi_1(X, p)$.

(b) Conversely, suppose that $[\gamma] = aba^{-1}b^{-1}$ for some $a, b \in \pi_1(X, p)$. Show that γ extends to a map $\Gamma : \Sigma \to X$.

(c) Replace Σ by the Möbius strip M, and find some algebraic condition on $[\gamma]$ that replaces $aba^{-1}b^{-1}$. That is, find some condition such that a map $\gamma : \partial M \to X$ extends to a map $\Gamma : M \to X$ if and only if $[\gamma]$ satisfies the condition.

(6) A *topological group* is a space X with a group structure such that both the multiplication map $X \times X \to X$ and the inversion map $X \to X$ are continuous.

(a) If G is a topological group with identity e, then if f and g are loops based at e (*ie.* paths that start and end at e), we can define the loop $f * g$ by setting $(f * g)(t) = f(t)g(t)$, using the group multiplication. Show that $[f * g] = [f \cdot g]$ as elements of $\pi_1(G, e)$ (where $*$ was just defined using the group multiplication, and $\cdot$ is the usual concatenation).

(b) Show that $\pi_1(G, e)$ is Abelian for topological groups G (even if G is not an Abelian group!).