3. Projective and affine planes

This chapter and the following two are in a relationship similar to the three parts of Chapter 2: the combinatorial part will be covered here, automorphisms (collineations) follow in Chapter 4, and constructions in Chapter 5.

Section 3.1, however, does not quite fit into this scheme. In that section we collect results from the general theory of projective planes, not depending on finiteness assumptions but relevant for later applications in the finite case. Section 3.1 also contains a few selected results on collineations and constructions of (not necessarily finite) projective and affine planes.

Section 3.2 is concerned with different systems of axioms for finite planes, with the connections between finite planes, certain sets of permutations, nets, and Latin squares, and with special substructures of finite planes, primarily subplanes, arcs, and ovals.

In Section 3.3, the rather few known results on dualities and polarities of finite planes are collected.

Section 3.4, finally, presents results on projectivities in projective planes. We include again some theorems for infinite planes, but the main part of 3.4 is concerned with results characterizing the finite desarguesian plane $\mathbf{P}(q) = \mathbf{P}_1(2, q)$ by properties of its group of projectivities.

3.1 General results

Projective and affine planes were defined in Section 1.4; we give equivalent definitions now. A *projective plane* is an incidence structure of points and lines satisfying the following conditions:

(1) *To any two distinct points, there exists a unique line incident with both of them.*

If $p \neq q$, then the unique line joining p and q is denoted by pq or qp.[1]

(2) *To any two distinct lines, there exists a unique point incident with both of them.*

---

[1] This is a more convenient notation than $p + q$, which was used in Section 1.4 because of the connection with vector space addition.

P. Dembowski, *Finite Geometries*

If $L \neq M$, then the unique common point of L and M is denoted by LM or ML.
(3) *There exist four points of which no three are incident with the same line.*

Such a set of four points is called a *quadrangle.* A set of points is *collinear* if there exists a line incident with every point of the set, and dually a set of lines is *concurrent* if there is a common point of all lines in the set. Hence a quadrangle may be defined as a set of four points no three of which are collinear, and dually a *quadrilateral* is a set of four lines no three of which are concurrent.

Conditions (1) and (2) are dual to each other, and together with (3) they imply the dual of (3): there exist quadrilaterals. Thus the theory of projective planes is self-dual in the sense that the dual structure $\bar{\mathbf{P}}$ (defined in Section 1.1) of an arbitrary projective plane $\mathbf{P}$ is again a projective plane. We shall see later that $\mathbf{P}$ and $\bar{\mathbf{P}}$ need not be isomorphic.

An *affine plane* is an incidence structure of points and lines satisfying (1) and the following conditions:

(4) *To any non-incident point-line pair p, L, there exists a unique line through p which has no point in common with L.*
(5) *There exist three non-collinear points.*

Such a set of three points is a *triangle,* and dually a set of three non-concurrent lines is a *trilateral.*

The dual of (5) is easily proved in any affine plane, but the dual of (4) is clearly false. Hence the dual of an affine plane is never an affine plane.[1]

In an arbitrary affine plane $\mathbf{A}$, define a relation $\parallel$ of *parallelism* among the lines as follows:

$$(6) \qquad L \parallel M \quad \textit{if and only if} \quad L = M \quad \textit{or} \quad [L, M] = 0.$$

The notation is that of (1.1.6) again. It is easily seen that $\parallel$ is an equivalence relation; the classes of this relation are called *ideal points.* If incidence in $\mathbf{A}$ is retained, if the ideal points are defined to be incident with exactly those lines of $\mathbf{A}$ which are contained in them, and if an *ideal line* W is introduced which is incident with all ideal points and no point of $\mathbf{A}$, then the extended incidence structure so defined is a projective plane $\mathbf{P}$, and $\mathbf{A}$ is the external structure $\mathbf{P}^W$ of $\mathbf{P}$ with respect to the ideal line W [see Section 1.1 for definitions]. Conversely, if L is

---

[1] Some authors, however, speak of the "dual" $\tilde{\mathbf{A}}$ of an affine plane $\mathbf{A} = \mathbf{P}^W$ in the following sense: Consider the dual $\bar{\mathbf{P}}$ of $\mathbf{P}$, which is a projective plane. Then $\tilde{\mathbf{A}} = (\bar{\mathbf{P}})^L$, where L is a line of $\bar{\mathbf{P}}$ through the point W of $\bar{\mathbf{P}}$. The line L of $\bar{\mathbf{P}}$ is usually unambiguously defined.

any line of a projective plane **P**, then $\mathbf{P}^L$ is an affine plane **A**, and **P** can be interpreted as constructed from **A** by means of ideal elements, as described above. On the other hand, if L, M are two distinct lines of **P**, then the affine planes $\mathbf{P}^L$ and $\mathbf{P}^M$ need not be isomorphic.

These considerations show that the theories of projective and affine planes are closely interrelated: an affine plane is essentially the same as a projective plane with a distinguished line. In some investigations, the projective point of view is more appropriate, and in others the affine point of view is. We shall encounter both situations later on.

A *subplane* of a projective plane **P** is a subset **S** of points and lines which is itself a projective plane, relative to the incidence relation given in **P**. This means that

(7) $p \in \mathbf{S}$ *and* $q \in \mathbf{S}$ *implies* $pq \in \mathbf{S}$,

(8) $L \in \mathbf{S}$ *and* $M \in \mathbf{S}$ *implies* $LM \in \mathbf{S}$,

(9) **S** *contains a quadrangle.*

Subsets **S** which satisfy (7) and (8) but not necessarily (9) will prove to be of interest later on. We call such subsets *closed*;[1] hence subplanes are those closed subsets which satisfy (9).

Intersections of closed subsets are again closed subsets. The closed subset *generated* by the arbitrary subset **S** of the projective plane **P** is defined as the intersection of all closed subsets containing **S**, and denoted by $\langle\mathbf{S}\rangle$, i.e.

$$\langle\mathbf{S}\rangle = \bigcap_{\substack{\mathbf{S} \subseteqq \mathbf{C} \\ \mathbf{C}\ \text{closed}}} \mathbf{C}. \tag{10}$$

It is easily verified that $\mathbf{S} \to \langle\mathbf{S}\rangle$ is a closure operation, i.e.

$$\langle\langle\mathbf{S}\rangle\rangle = \langle\mathbf{S}\rangle, \quad \mathbf{S} \subseteqq \langle\mathbf{S}\rangle, \quad \text{and} \quad \mathbf{S} \subseteqq \mathbf{T} \to \langle\mathbf{S}\rangle \subseteqq \langle\mathbf{T}\rangle.$$

Hence[2] the closed subsets of an arbitrary projective plane **P** form a complete lattice $\mathscr{C}(\mathbf{P})$, and the subplanes, together with the empty set, form a complete lattice $\mathscr{S}(\mathbf{P})$. Little seems to be known about these lattices in general.[3]

A *prime plane* is a projective plane **P** which does not possess proper subplanes, i.e. one for which $|\mathscr{S}(\mathbf{P})| = 2$. Clearly,

1. *A projective plane is a prime plane if and only if it is generated by each one of its quadrangles.*

---

[1] Closed subsets which are not subplanes are sometimes called "degenerate subplanes". For a complete classification of all possible types of these, see HALL 1943, or PICKERT 1955, p. 13.

[2] cf. BIRKHOFF 1948, p. 49—50.

[3] An example in HALL 1943 shows that $\mathscr{S}(\mathbf{P})$ need not be modular. Note that $\mathscr{S}(\mathbf{P})$ need not be a sublattice of $\mathscr{C}(\mathbf{P})$ in the sense of BIRKHOFF 1948, p. 19.

A subplane of a projective plane **P** is called *minimal* if it is a prime plane, i.e. an atomic element[1] of $\mathcal{S}(\mathbf{P})$. A *maximal closed subset,* or a *maximal subplane,* is an anti-atomic element $\neq \emptyset$ of $\mathcal{C}(\mathbf{P})$ or $\mathcal{S}(\mathbf{P})$, respectively. A closed subset **C** of **P** is called a *Baer subset*[2], or a *Baer subplane* in case it is a subplane, if it satisfies the following conditions:

(11) *Every point of* **P** *is incident with a line of* **C**.

(12) *Every line of* **P** *is incident with a point of* **C**.

Clearly every Baer subset is a maximal closed subset, but there are examples of maximal subplanes which are not Baer subplanes. On the other hand:

2. *A closed subset which is not a subplane is maximal if and only if it is a Baer subset; these Baer subsets are precisely the sets*

$$\mathbf{B}(p, L) = \{p, L\} \cup (p) \cup (L)$$

for some point-line pair p, L [*notation as in* (1.1.5)].

This is easily verified; the cases $p \mathrel{\mathrm{I}} L$ and $p \not\mathrel{\mathrm{I}} L$ are both admissible. Baer subsets and subplanes will be encountered frequently in the finite case.

A *subplane of an affine plane* $\mathbf{A} = \mathbf{P}^W$ is a subset **S** of points and lines which is itself an affine plane, relative to the incidence given in **A**, such that

(13) *If two lines are parallel in* **S**, *then they are also parallel in* **A**.

One could, of course, define affine subplanes without this condition,[3] but then there would be no connection between the subplanes of $\mathbf{A} = \mathbf{P}^W$ and those of **P**. Condition (13) guarantees such a connection:

3. *The subplanes of the affine plane* $\mathbf{A} = \mathbf{P}^W$ *are precisely the affine planes* $\mathbf{S}^W$, *where* **S** *ranges over all projective subplanes of* **P** *which contain the line* W.

It will be clear what we mean by minimal, maximal, and Baer subplanes of affine planes: we require that these subplanes, when interpreted as projective planes, contain the ideal line W.

We turn now to *collineations* of projective planes. These are automorphisms as defined in Section 1.2, i.e. incidence preserving per-

---

[1] Atomic and anti-atomic elements need not exist in an arbitrary lattice $\mathcal{S}(\mathbf{P})$, but clearly such elements exist if **P** is finite.

[2] After BAER 1946b, who first realized the importance of these subsets as systems of fixed elements with respect to certain collineations. This will be discussed in detail further below.

[3] Cf. OSTROM & SHERK 1964; RIGBY 1965.

mutations which map points onto points and lines onto lines.[1]) The group of all collineations of a projective plane **P** will be denoted by Aut**P**, as in Section 1.2; a *collineation group* of **P** is any subgroup of Aut**P**. The full collineation group of an affine plane $\mathbf{A} = \mathbf{P}^W$ is essentially identical with the stabilizer of W in Aut**P**.

Our main interest here lies in central collineations of projective planes, as defined in Section 1.4. A *center* of a collineation $\alpha \in \operatorname{Aut}\mathbf{P}$ is a point c such that $X\alpha = X$ for all lines $X \mathrm{I}\, c$. Dually, an *axis* of α is a line A with $x\alpha = x$ for all $x \mathrm{I} A$.

4. *A collineation α of a projective plane has a center if and only if it has an axis. If $\alpha \neq 1$, then the center and the axis of α are unique.*

Proof: Pickert 1955, p. 62—65; see also **1.4.8** and **1.4.9** above. As in 1.4, we define *elations* and *homclogies* as central collineations with incident and non-incident center-axis pairs, respectively. A simple but useful fact about central collineations is the following:

5. *Let α be a collineation with center c and axis A, and suppose that $c \neq p \,\mathrm{I}\, A$ for some point p. Then every subplane containing c, A, p, $p\alpha$ is left invariant by α.*

For the proof, see Lüneburg 1964c, p. 446—447.

The set $\mathbf{F} = \mathbf{F}(\alpha)$ of the elements fixed by the central collineation $\alpha \neq 1$, with center c and axis A, is the Baer subset $\mathbf{B}(c, A)$ described in **2**. A collineation whose fixed elements form a Baer subset will be called *quasicentral*. Hence a collineation which is quasicentral but not central has a Baer subplane of fixed elements. Such collineations will be of interest in the finite case; here we note only that

6. *Involutorial collineations are quasicentral.*

For the easy proof, see Baer 1946b, p. 275.

Let Γ be an arbitrary collineation group of the projective plane **P**, and let c, p be points and A, L lines of **P**. Then each of the following sets is a subgroup of Γ:

$$(14)\qquad \left\{\begin{aligned} \Gamma(c, A) &= \{\gamma \in \Gamma : \gamma \text{ has center } c \text{ and axis } A\},\\ \Gamma(L, A) &= \bigcup_{x \mathrm{I} L} \Gamma(x, A),\ \Gamma(c, p) = \bigcup_{X \mathrm{I} p} \Gamma(c, X)\\ \Gamma(A) &= \bigcup_{x} \Gamma(x, A),\ \Gamma(c) = \bigcup_{X} \Gamma(c, X). \end{aligned}\right.$$

Here the last two unions are to be taken over all points x and all lines X of **P**, respectively. It follows from **4** that the unions in (14) are in

---

[1]) The requirement of section 1.2, that inverses be incidence-preserving [cf. footnote [1]) of p. 8] is easily proved for projective and affine planes.

fact partitions. The groups listed in (14) consist of central collineations, either with common axis or with common center. The product of two central collineations is, in fact, usually not central unless they have the same center or axis:

7. *Let* $1 \neq \alpha \in \Gamma(a, A)$ *and* $1 \neq \beta \in \Gamma(b, B)$, *and suppose that* $a \neq b$ *and* $A \neq B$. *Then:*

(a) *A fixed point of* $\alpha\beta$ *is either* AB *or incident with* ab.

(b) $\alpha\beta$ *is a central collineation if and only if* α *and* β *are homologies such that*

(15) $\quad a \,\mathrm{I}\, B, \; b \,\mathrm{I}\, A,$ *and* $x\alpha = x\beta^{-1}$ *for every* $x \,\mathrm{I}\, ab$.

If this is the case, then $\alpha\beta$ *is a homology with center* AB *and axis* ab.

Proof[1]: That (**a**) holds is easily verified; in fact if $AB \neq x \not\mathrm{I}\, ab$, then x, $x\alpha$ and $x\alpha\beta$ are non-collinear points. Hence if $\alpha\beta$ is central, its axis must be $C = ab$, whence $x\alpha = x\beta^{-1}$ for all $x \,\mathrm{I}\, C$. Putting $x = a$ and using the fact that $a \neq b$, we get $a \,\mathrm{I}\, B$, and $b \,\mathrm{I}\, A$ follows in the same fashion. As $A \neq B$, at least one of α, β, say α, is a homology. If β were an elation, then $B = ab = C$, and $\alpha = (\alpha\beta)\beta^{-1}$ would also have axis $A = C$, against the hypothesis. The remainder is clear.

It follows from **7** that if a collineation group Γ consists of central collineations with neither the same center nor the same axis, then Γ is the non-cyclic group of order 4, and its non-trivial elements are three involutorial homologies whose centers and axes are the vertices, and opposite sides, of a triangle. The first part of the next result shows that this situation can actually occur:

8. *Let* α *and* β *be two involutorial homologies, in* $\Gamma(a, A)$ *and* $\Gamma(b, B)$, *respectively, and put* $\gamma = \alpha\beta$.

(a) *If* $a \,\mathrm{I}\, B \neq A \,\mathrm{I}\, b \neq a$, *then* γ *is an involutorial homology in* $\Gamma(AB, ab)$.

(b) *If* $a \neq b$ *and* $A = B$, *then* γ *is an elation in* $\Gamma(A(ab), A)$.

Proof: Ostrom 1956, Lemmas 4 and 6.[2] Note that in case (**a**), both $\Gamma(a, A)$ and $\Gamma(b, B)$ contain no involution $\neq \alpha$ or β, respectively. In case (**b**), we can draw a similar conclusion:

9. *If the group* $\Gamma(A)$ *contains nontrivial homologies with different centers, then, for any point* $c \not\mathrm{I}\, A$, *the group* $\Gamma(c, A)$ *contains at most one involution.*

---

[1] This result is well known, but the author has been unable to locate a convenient reference.

[2] Both results are, however much older. For example, a proof of (**b**) was given by Baer 1944, p. 103.

For the proof, the following simple but important fact is needed:

10. *If* α *is in the normalizer of* Γ *in* $\mathrm{Aut}\,\mathbf{P}$, *then*

$$\alpha^{-1}\,\Gamma(c, A)\,\alpha = \Gamma(c\,\alpha, A\,\alpha) \tag{16}$$

for any point-line pair (c, A) *of* $\mathbf{P}$.

The proof of **10** is straightforward. We prove **9**: Let ϱ and σ be involutorial homologies in $\Gamma(A)$ and assume that they have the same center $p\,\bar{\mathrm{I}}\,A$. Then also $\varrho\,\sigma \in \Gamma(p, A)$. On the other hand, there exists $\alpha \neq 1$ in $\Gamma(A)$ with center $\neq p$, and $\tau = \alpha^{-1}\varrho\alpha$ is, by **10**, an involutorial homology with axis A and center $p\,\alpha \neq p$. But then **8b** shows that $\varrho\,\tau$ and $\tau\,\sigma$ are both elations, whence $\varrho\,\sigma = \varrho\,\tau \cdot \tau\,\sigma$ is likewise an elation. This is compatible with $\varrho\,\sigma \in \Gamma(p, A)$ only if $\varrho\,\sigma = 1$ or $\varrho = \sigma$.

If c and A are fixed by Γ, then $\Gamma(c, A)$ is a normal subgroup of Γ, because of **10**. It follows that

$$\Gamma(A, A) \lhd \Gamma(A) \quad \textit{and} \quad \Gamma(c, c) \lhd \Gamma(c) \tag{17}$$

for any A, c. Also, $\Gamma(c, A) \lhd \Gamma(A)$ and $\Gamma(c, A) \lhd \Gamma(c)$ whenever $c\,\mathrm{I}\,A$. This can also be concluded from (17) and the following result:

11. *If* $\Gamma(A, A)$ *contains nontrivial elations with different centers* (*on* A), *then* $\Gamma(A, A)$ *is abelian.*

The proof is again easy, see Pickert 1955, p. 199. By a very similar argument, one may in fact prove more about commuting central collineations:

12. *Let* (a, A) *and* (b, B) *be two distinct point-line pairs in* $\mathbf{P}$ *and* $1 \neq \alpha \in \Gamma(a, A)$, $1 \neq \beta \in \Gamma(b, B)$. *Then* $\alpha\,\beta = \beta\,\alpha$ *if and only if* $a\,\mathrm{I}\,B$ *and* $b\,\mathrm{I}\,A$.

Note the connection with **7**.

We shall now be interested in the case where $\alpha\,\beta \neq \beta\,\alpha$. Hence let (a, A) and (b, B) be distinct point-line pairs of $\mathbf{P}$ such that

$$a\,\bar{\mathrm{I}}\,B \quad \text{or} \quad b\,\bar{\mathrm{I}}\,A. \tag{18}$$

For any $\alpha \neq 1$ in $\Gamma(a, A)$ and $\beta \neq 1$ in $\Gamma(b, B)$ we consider the mappings

$$\alpha^* : \xi \to \xi^{-1}\,\alpha^{-1}\,\xi\,\alpha \quad \text{and} \quad \beta^* : \eta \to \beta^{-1}\,\eta^{-1}\,\beta\,\eta \tag{19}$$

from $\Gamma(b, B)$ or $\Gamma(a, A)$, respectively, into Γ. It follows from **12** and (18) that both α^* and β^* are one-one. Now **10** shows that

$$\beta^{-1}\,\alpha^{-1}\,\beta\,\alpha \in \Gamma(a\,\beta, A\,\beta)\,\Gamma(a, A) \cap \Gamma(b, B)\,\Gamma(b\,\alpha, B\,\alpha).$$

Hence if $a\,\mathrm{I}\,B$, which implies $B\,\alpha = B$ and $a\,\beta = a$, then

$$\beta^{-1}\,\alpha^{-1}\,\beta\,\alpha = \beta^{\alpha^*} = \alpha^{\beta^*} \in \Gamma(a) \cap \Gamma(B) = \Gamma(a, B).$$

As α^* and β^* are one-one, we can conclude:

13. *Let* $a \,\mathrm{I}\, B$ *and* $b \not\mathrm{I}\, A$.

(**a**) *If* $\Gamma(a, A) \neq 1$, *then* $|\Gamma(a, B)| \geqq |\Gamma(b, B)|$.
(**b**) *If* $\Gamma(b, B) \neq 1$, *then* $|\Gamma(a, B)| \geqq |\Gamma(a, A)|$.

Thus if $\Gamma(a, A)$ or $\Gamma(b, B)$ is nontrivial, then so is $\Gamma(a, B)$. In a special case, we can say more:

14. *If* $B \neq A \,\mathrm{I}\, a \,\mathrm{I}\, B \,\mathrm{I}\, b \neq a$ *and* $\Gamma(a, A) \neq 1 \neq \Gamma(b, B)$, *then* $\Gamma(a, B)$ *contains subgroups* (*which may coincide*) *isomorphic to* $\Gamma(a, A)$ *and* $\Gamma(b, B)$, *respectively.*

Proof. One verifies first that the mapping α^* from $\Gamma(b, B)$ into Γ satisfies

$$(\xi\, \eta)^{\alpha^*} = \xi^{\alpha^*} \eta^{\alpha^*}$$

if and only if $\Gamma(b, B)^{\alpha^*}$ is in the centralizer of $\Gamma(b, B)$ in Γ. But under the hypothesis of **14**, we have $\Gamma(b, B)^{\alpha^*} \subseteqq \Gamma(a, B) \neq 1$, and $\Gamma(B, B)$ is abelian by **11**, so that this centralizer condition is satisfied. Hence α^* is an isomorphism into $\Gamma(a, B)$, and $\Gamma(a, B)$ contains an isomorphic copy of $\Gamma(b, B)$. The remainder of **14** follows from a dual argument.1)

A central collineation is uniquely determined by the image of any one of its non-fixed points. More precisely:

15. *Let* (c, A) *be a point-line pair in* **P** *and* x, y *two points such that* $x \neq c \neq y \not\mathrm{I}\, A \not\mathrm{I}\, x$ *and* $cx = cy$. *Then there is at most one* $\gamma \in \Gamma(c, A)$ *with* $x\gamma = y$.

Proof: Pickert 1955, p. 66.

The group Γ will be called (c, A)-*transitive* if the "at most" in **15** can be replaced by "exactly", in other words if $\Gamma(c, A)$ is transitive on the non-fixed points of any line $\neq A$ through c. (If this is so for one such line, then it can be proved for all others as well; see Pickert 1955, p. 66.) Also, (c, A)-transitivity may be defined dually by transitivity of $\Gamma(c, A)$ on the non-fixed lines through any point $\neq c$ on A.

Next, we say that a projective plane **P** is (c, A)-*transitive* if its full collineation group Aut**P** is (c, A)-transitive. This concept, due to Baer 1942, has proved to be a very useful classifying principle for projective planes; we shall discuss this now at some length. First, there is a close connection with a special case of Desargues' theorem. We say that a projective plane **P** is (p, L)-*desarguesian* if every central couple2) $(p_1\, p_2\, p_3)$, $(p'_1, p'_2, p'_3,)$ of triangles with $p_i\, p'_i \,\mathrm{I}\, p, (i = 1, 2, 3)$ and $(p_1\, p_2)\, (p'_1\, p'_2) \,\mathrm{I}\, L \,\mathrm{I}\, (p_2\, p_3)\, (p'_2\, p'_3)$ is axial. It is not difficult to prove that

---

1) Compare here Hering 1963, p. 156. We remark that the mapping $\alpha \to \alpha^*$ is never a homomorphism; in fact $(\alpha\beta)^* \neq \alpha^* \beta^*$ for all $\alpha, \beta \in \Gamma(b, B)$.

2) For the definition of central and axial couples of triangles, see Section 1.4.

16. *$\mathbf{P}$ is (p, L)-transitive if and only if $\mathbf{P}$ is (p, L)-desarguesian.*

A proof appears in PICKERT 1955, pp. 76—78; three other equivalent properties are also given there.

We proceed now to a complete classification of the various possibilities for distinct (p, L)-transitivities in an arbitrary collineation group. The following preliminary results are basic for this classification.

17. *If Γ is (c, A)-transitive and if γ is in the normalizer of Γ in* Aut$\mathbf{P}$, *then Γ is also $(c\gamma, A\gamma)$-transitive.*

This follows immediately from **10**. We say that Γ is *(L, A)-transitive* if Γ is (c, A)-transitive for every $c \mathrm{I} L$; dually, Γ is *(c, p)-transitive* if Γ is (c, A)-transitive for every $A \mathrm{I} p$. Also, (L, A)- and (c, p)-transitivity for $\mathbf{P}$ is defined as (L, A)- resp. (c, p)-transitivity for Aut$\mathbf{P}$.

18. *Suppose that $\mathbf{P}$ has more than three points per line. If Γ is (c, A)- and (c', A)-transitive for two distinct points c, c', then Γ is also (cc', A)-transitive. Dually, (c, A)- and (c, A')-transitivity for $A \neq A'$ implies (c, AA')-transitivity.*

Proof: BAER 1942; PICKERT 1955, pp. 67—68. Note that if $\Gamma = \mathrm{Aut}\mathbf{P}$, then **18** holds also for the plane $\mathbf{P}(2)$ with only three points per line. This is a first instance of the situation that stronger conclusions can be drawn from certain (p, L)-transitivities of $\mathbf{P}$ than from those of Γ. A more interesting case for this situation is:

19. *A projective plane $\mathbf{P}$ is (p, q)-transitive if and only if it is (q, p)-transitive. Dually, (L, A)- and (A, L)-transitivity for $\mathbf{P}$ are equivalent.*

(GINGERICH 1945; PICKERT 1955, p. 103.) Clearly, **19** ceases to be true if "$\mathbf{P}$" is replaced by "Γ".[1]

Now we present the classification mentioned above.

20. *For any collineation group Γ of a projective plane $\mathbf{P}$ with more than five*[2] *points per line, define*

$$\mathbf{T} = \mathbf{T}(\Gamma) = \{(x, X) \vdots \Gamma \text{ is } (x, X)\text{-transitive}\}. \tag{20}$$

---

[1] It should be mentioned that while **16—18** are quite elementary, the simplest proof of **19** seems to require the use of coordinates. For this, see p. 127 and result **22e** below.

[2] This hypothesis, more restrictive than that of **18**, is essential. For example, if π is a unitary polarity of $\mathbf{P}(4)$, then $\Gamma_0(\pi) \cong PGU_3(4)$ [cf. p. 47] is (p, L)-transitive if and only if p is non-absolute and $L = p\pi$. This corresponds to none of the types I.1—VII.2 below.

Then $\mathbf{T}$ *is of one of the following types:*

I.1. $\mathbf{T} = \emptyset$.

I.2. $\mathbf{T} = \{(c, A)\}, \quad c \not\mathrm{I} A$.

I.3. $\mathbf{T} = \{(c, A), (c', A')\}, \quad c \not\mathrm{I} A \,\mathrm{I}\, c' \not\mathrm{I} A' \,\mathrm{I}\, c$.

I.4. $\mathbf{T} = \{(a, A), (b, B), (c, C)\}$, (*vertices and opposite sides of a triangle*).

I.5. $\mathbf{T} = \{(x, x^\sigma p) : x \,\mathrm{I}\, L\}$, $p \not\mathrm{I} L$, σ *a fixed point free involution of* (L).

I.6. $\mathbf{T} = \{(x, x^\sigma) : p \neq x \,\mathrm{I}\, L\}$, $p \,\mathrm{I}\, L$, σ *one-one from* $(L) - \{p\}$ *onto* $(p) - \{L\}$.

I.7. $\mathbf{T} = \{(p, L)\} \cup \{(x, x^\sigma p) : x \,\mathrm{I}\, L\}$, p, L, σ *as in* I.5.

I.8. $\mathbf{T} = \{(x, x^\pi)\}$, π *a polarity of* $\mathbf{P}$, *without absolute points.*

II.1. $\mathbf{T} = \{(c, A)\}, \quad c \,\mathrm{I}\, A$.

II.2. $\mathbf{T} = \{(c, A), (c', A')\}, \quad A' \neq A \,\mathrm{I}\, c' \neq c = A\,A'$.

II.3. $\mathbf{T} = \{(p, L)\} \cup \{(x, x^\sigma) : p \neq x \,\mathrm{I}\, L\}$, $\quad p, L, \sigma$ *as in* I.6.

II.4a. $\mathbf{T} = \{(x, A) : x \,\mathrm{I}\, L\}, \quad L \neq A$.

II.5a. $\mathbf{T} = \{(p, L)\} \cup \{(x, A) : x \,\mathrm{I}\, L\}, \quad L \neq A \,\mathrm{I}\, p \neq AL$.

II.4b *and* II.5b *are dual to* II.4a *and* II.5a, *respectively.*

$$\left.\begin{array}{ll} \text{III.1.} & \mathbf{T} = \{(x, p\,x) : x \,\mathrm{I}\, L\}, \\ \text{III.2.} & \mathbf{T} = \{(p, L)\} \cup \{(x, p\,x) : x \,\mathrm{I}\, L\}, \\ \text{III.3.} & \mathbf{T} = \{(x, p\,y) : x, y \,\mathrm{I}\, L\}, \\ \text{III.4.} & \mathbf{T} = \{(p, L)\} \cup \{(x, p\,y) : x, y \,\mathrm{I}\, L\}, \end{array}\right\} p \not\mathrm{I} L.$$

IVa.1. $\mathbf{T} = \{(x, A) : x \,\mathrm{I}\, A\}$,

IVa.1′. $\mathbf{T} = \{(x, A) : x \,\mathrm{I}\, A\} \cup \{(p, Y) : Y \,\mathrm{I}\, q\}, \quad p \neq q;\ p, q \,\mathrm{I}\, A$.

IVa.2. $\mathbf{T} = \{(x, A) : x \,\mathrm{I}\, A\} \cup \{(p, Y) : Y \,\mathrm{I}\, q\} \cup \{(q, Z) : Z \,\mathrm{I}\, p\}$, $\quad p \neq q;\ p, q \,\mathrm{I}\, A$.

IVa.3. $\mathbf{T} = \{(x, Y) : x \,\mathrm{I}\, A;\ Y \,\mathrm{I}\, x^\sigma\}$, $\quad \sigma$ *a fixed point free involution of* (A).

IVa.4. $\mathbf{T} = \{(x, A)$, *all* $x\}$.

IVa.5. $\mathbf{T} = \{(x, A)$, *all* $x\} \cup \{(p, Y) : Y \,\mathrm{I}\, q\}, \quad p \neq q;\ p, q \,\mathrm{I}\, A$.

IVa.6. $\mathbf{T} = \{(x, A)$, *all* $x\} \cup \{(p, Y) : Y \,\mathrm{I}\, q\} \cup \{(q, Z) : Z \,\mathrm{I}\, p\}$, $p \neq q;\ p, q \,\mathrm{I}\, A$.

IVa.7. $\mathbf{T} = \{(x, A)$, *all* $x\} \cup \{(y, Y) : y \,\mathrm{I}\, A;\ Y \,\mathrm{I}\, y^\sigma\}$, σ *as in* IVa.3.

IVb.1.—IVb.7. *are dual to* IVa.1.—IVa.7., *respectively.*

$$\left.\begin{array}{ll}
\text{V.1.} & \mathbf{T} = \{(x, A) : x\,\mathrm{I}\,A\} \cup \{(c, Y) : Y\,\mathrm{I}\,c\},\\
\text{V.2.} & \mathbf{T} = \{(z, Z) : z\,\mathrm{I}\,A; Z\,\mathrm{I}\,c\},\\
\text{V.3a.} & \mathbf{T} = \{(x, A),\ \textit{all } x\} \cup \{(c, Y) : Y\,\mathrm{I}\,c\},\\
\text{V.4.} & \mathbf{T} = \{(x, A),\ \textit{all } x\} \cup \{(c, Y),\ \textit{all } Y\},\\
\text{V.5a.} & \mathbf{T} = \{(x, A),\ \textit{all } x\} \cup \{(y, Y) : y\,\mathrm{I}\,A;\ Y\,\mathrm{I}\,c\},\\
\text{V.6.} & \mathbf{T} = \{(x, A),\ \textit{all } x\} \cup \{(c, Y),\ \textit{all } Y\}\\
 & \qquad\qquad \cup\ (z, Z) : z\,\mathrm{I}\,A; Z\,\mathrm{I}\,c\},
\end{array}\right\}\ c\,\mathrm{I}\,A.$$

V.3b *and* V.5b *are dual to* V.3a *and* V.5a, *respectively.*

$$\left.\begin{array}{ll}
\text{VIa.1.} & \mathbf{T} = \{(x, Y) : A\,\mathrm{I}\,x\,\mathrm{I}\,Y\},\\
\text{VIa.2.} & \mathbf{T} = \{(x, A),\ \textit{all } x\} \cup \{(y, Y) : A\,\mathrm{I}\,y\,\mathrm{I}\,Y\},\\
\text{VIa.3.} & \mathbf{T} = \{(x, Y),\ \textit{all } Y; x\,\mathrm{I}\,A\},\\
\text{VIa.4.} & \mathbf{T} = \{(x, A),\ \textit{all } x\} \cup \{(y, Y),\ \textit{all } Y; y\,\mathrm{I}\,A\},
\end{array}\right\}\ \begin{array}{l}\textit{for some}\\ \textit{line } A.\end{array}$$

VIb.1—VIb.4 *are dual to* VIa.1—VIa.4, *respectively.*

VII.1. $\mathbf{T} = \{(x, X) : x\,\mathrm{I}\,X\}$,

VII.2. $\mathbf{T} = \{(x, X),\ \textit{all } x, X\}$.

This theorem is proved by multiple application of **17** and **18** above, in the spirit of LENZ 1954 and BARLOTTI 1957b.[1] Consequently, we refer to the 53 possibilities for $\mathbf{T}(\Gamma)$, listed in **20**, as the *Lenz-Barlotti types for collineation groups* of projective planes.

With the exception of I.8, there actually exist collineation groups Γ for each of the types in **20**.[2] In many cases it can be proved, however, that Γ cannot be the full collineation group Aut$\mathbf{P}$. This leads to the problem of determining the possibilities for $\mathbf{T}(\mathrm{Aut}\mathbf{P})$; these are called the *Lenz-Barlotti types for projective planes* $\mathbf{P}$. The possibilities are much more limited here; this follows from **19** and other considerations, of which many will be at least outlined further below. The following

---

[1] These authors were only interested in the case $\Gamma = \mathrm{Aut}\mathbf{P}$; here **19** is also applicable and leads to the exclusion of many of the types listed in **20**. Furthermore, Lenz only determined the subsets of *incident* point-line pairs in $\mathbf{T}(\mathrm{Aut}\mathbf{P})$. There are only seven such types, referred to by the Roman numerals in **20**. Barlotti's work consisted in refining Lenz's classification so as to include also non-incident center-axis pairs. (Incidentally, the reason that we have called one of the types IVa. 1′ is only that we wanted to retain Barlotti's numbering; there is no mathematical reason for it.) Another refinement of the Lenz-Barlotti classification, taking into account also certain correlations of the plane (cf. Section 3.3), was given by JÓNSSON 1963.

[2] The reason for including I.8 here is that the nonexistence proof for such collineation groups requires rather more than **17** and **18**; cf. **4.3.32** below. For the finite case, see also **3.3.2**.

table represents the present state of knowledge on the existence problem for Lenz-Barlotti types (of groups and of planes):

Table 1: *Lenz-Barlotti types*

Type	Finite case		Infinite case	
	Group exists	Plane exists	Group exists	Plane exists
I.1	yes	yes; cf. 5.4	yes	yes (Hilbert 1899)
I.2	yes	?	yes	yes (Spencer 1960)
I.3	yes	?	yes	yes (Yaqub 1961 b)
I.4	yes	?	yes	yes (Naumann 1954)
I.5	yes; $n \leqq 9$ cf. 4.3	no	no	no
I.6	yes; $n \leqq 4$	no	?	?
I.7	yes; $n \leqq 5$	no	no	no
I.8	no	no	no	no
II.1	yes	yes; cf. 5.4	yes	yes (Spencer 1960)
II.2	yes	?	yes	yes[1]
II.3	yes; $n \leqq 4$ (cf. 4.3)	no	no	no (Spencer 1960)
II.4 a	yes	no by **19**	yes	no by **19**
II.5 a	yes	no	yes	no
III.1	yes	?	yes	yes (Yaqub 1961 a)
III.2	yes	no; cf. 4.3	yes	yes (Moulton 1902)
III.3	yes	no by **19**	yes	no by **19**
III.4	yes	no	yes	no
IV a.1	yes	yes; cf. 5.2	yes	yes
IV a.1′	yes	no, by **19**	yes	no, by **19**
IV a.2	yes	yes; cf. 5.2	yes	yes
IV a.3	yes; $n \leqq 9$ (cf. 4.3)	yes; $n = 9$	no	no
IV a.4	yes	no by **19**	yes	no by **19**
IV a.5	yes	no	yes	no
IV a.6	yes	no	yes	no
IV a.7	yes	no	yes	no
V.1	yes	yes; cf. 5.3	yes	yes
V.2	yes	no by **19**	yes	no by **19**
V.3 a	yes	no	yes	no
V.4	yes	no	yes	no
V.5 a	yes	no	yes	no
V.6	yes	no	yes	no
VI a.1	yes	no; cf. **22**	yes	no; cf. **22**
VI a.2	yes	no by **19**	yes	no by **19**
VI a.3	yes	no	yes	no
VI a.4	yes	no	yes	no
VII.1	yes	no; cf. **22**	yes	yes (Moufang 1933)
VII.2	yes	yes	yes	yes

---

[1] No example of this seems to have been published. For the following construction, the author is indebted to H. Salzmann. Call points the pairs (x, y) of real numbers, lines the point sets $L(a, b) = \{(x, (x^{\alpha} a^{\alpha})^{\alpha^{-1}} + b)\}$ and $L(c) = \{(c, y)\}$, where α is a non-identity order preserving permutation of the reals, and define incidence by set theoretical inclusion. This yields an affine plane, and the corresponding projective plane is of Lenz-Barlotti type II.2.

The main tool for the proofs of many of the results in Table 1 is the introduction of *coordinates* in a projective plane. We give a brief account of this now, following essentially HALL 1943. (See also HALL 1959, Section 20.3.)

Let o, e, u, v be an (ordered) quadrangle in the projective plane $\mathbf{P}$, put $W = u\,v$, and let $\mathbf{A}$ denote the affine plane $\mathbf{P}^W$. We consider a set $\mathfrak{T}$ with the same cardinality as a line of $\mathbf{A}$; for example, $\mathfrak{T}$ may consist of the lines $\neq W$ through u. Two distinct elements in $\mathfrak{T}$ are called 0 and 1. We set up a one-one correspondence between the points of $\mathbf{A}$ and the ordered pairs (x, y) of elements in $\mathfrak{T}$ as follows: The points $\neq u$ of ou are assigned the pairs $(x, 0), x \in \mathfrak{T}$, such that $o = (0, 0)$ and $(ou)\,(ev) = (1, 0)$. The points $\neq v$ of ov are then assigned the pairs $(0, y), y \in \mathfrak{T}$, by the following rule:

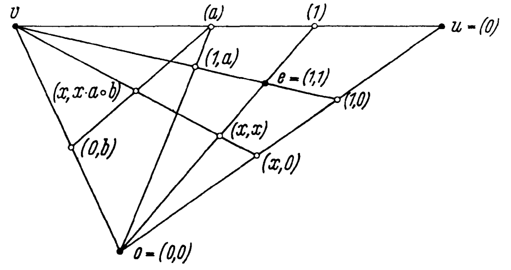

Fig. 2. Coordinates

$$(0, y) = (o\,v)\,\big(u[(o\,e)\,\{(y, 0)\,v\}]\big).$$

In particular, $(0, 1) = (o\,v)\,(u\,e)$. Finally, we put

$$(x, y) = [(x, 0)\,v]\,[(0, y)\,u]; \tag{21}$$

note that this assigns the pairs (x, x) to the points of the line oe, and in particular $e = (1, 1)$. Next, we label the points of the line W which are different from v:

$$(t) = [(0, 0)\,(1, t)]\,W. \tag{22}$$

Now we define a ternary operation in $\mathfrak{T}$ as follows (cf. Fig. 2):

$$(x, x \cdot a \circ b) = [(x, 0)\,v]\,[(a)\,(0, b)]. \tag{23}$$

In other words, the line through (a) and $(0, b)$ is represented by the equation $y = x \cdot a \circ b$. The axioms (1)—(3) for $\mathbf{P}$ imply the following properties of this operation:

$$(24)\quad \left\{\begin{array}{ll} \text{(a)} & x \cdot 0 \circ b = 0 \cdot x \circ b = b, \qquad \textit{for all } x, b \in \mathfrak{T}. \\ \text{(b)} & x \cdot 1 \circ 0 = 1 \cdot x \circ 0 = x, \qquad \textit{for all } x \in \mathfrak{T}. \\ \text{(c)} & \textit{Given } x, y, a \in \mathfrak{T}, \textit{ there is a unique } b \in \mathfrak{T} \\ & \quad \textit{such that } y = x \cdot a \circ b. \\ \text{(d)} & \textit{Given } x_i, y_i \in \mathfrak{T}\ (i = 1, 2) \textit{ with } x_1 \neq x_2, \\ & \quad \textit{there is a unique ordered pair } (a, b) \textit{ such that} \\ & \qquad\qquad y_i = x_i \cdot a \circ b\ (i = 1, 2). \\ \text{(e)} & \textit{Given } a_i, b_i \in \mathfrak{T}\ (i = 1, 2) \textit{ with } a_1 \neq a_2, \textit{ there is a} \\ & \quad \textit{unique } x \textit{ such that } x \cdot a_1 \circ b_1 = x \cdot a_2 \circ b_2. \end{array}\right.$$

Any set $\{0, 1, \ldots\}$ with a ternary operation satisfying (24) will be called here a *ternary field.*[1] Thus every ordered quadrangle o, e, u, v of a projective plane **P** gives rise to a ternary field $\mathfrak{T} = \mathfrak{T}(o, e, u, v)$, and:

21. *$\mathfrak{T}(o, e, u, v)$ and $\mathfrak{T}(o', e', u', v')$ are isomorphic if and only if there exists a collineation mapping o onto o', e onto e', u onto u', and v onto v'.*

Proof: Pickert 1955, p. 37—38.

Given any ternary field $\mathfrak{T}$, the incidence structure $\mathbf{A} = \mathbf{A}(\mathfrak{T})$ defined as follows is an affine plane: points of **A** are the ordered pairs (x, y) with $x, y \in \mathfrak{T}$, lines are the point sets

$$(25)\qquad \begin{aligned} L(a, b) &= \{(x, x \cdot a \circ b) \vdots\, x \in \mathfrak{T}\}, \\ L(c) &= \{(c, y) \vdots\, y \in \mathfrak{T}\}, \end{aligned}$$

and incidence is set theoretic inclusion. In the projective plane **P** corresponding to **A**, let $o = (0, 0)$, $e = (1, 1)$, $u = (0)$, and $v =$ ideal point of $L(0)$. Then the ternary field $\mathfrak{T}(o, e, u, v)$ is essentially identical with the given $\mathfrak{T}$. Thus there is a canonical correspondence between ternary fields and projective planes with distinguished ordered quadrangles.

In any ternary field $\mathfrak{T}$, *addition* and *multiplication* are defined as follows:

$$(26)\qquad \begin{aligned} a + b &= a \cdot 1 \circ b \\ a\, b &= a \cdot b \circ 0. \end{aligned}$$

---

[1] The more customary term is "planar ternary ring" (Hall 1943, 1959). The present terminology follows Pickert 1955 ("Ternärkörper"), the reason being that there are no proper homomorphisms between ternary fields, and hence no ideals. This follows from (24).

With respect to addition, $\mathfrak{T}$ is a loop with neutral element 0; this loop will be denoted by $\mathfrak{T}^+$. The set $\mathfrak{T} - \{0\}$ is a loop with respect to multiplication, with neutral element 1, which will be denoted by $\mathfrak{T}^\times$. A ternary field is *linear* if it satisfies

$$(27) \qquad x \cdot a \circ b = x a + b \qquad \textit{for all } x, a, b \in \mathfrak{T}.$$

It can be verified that linearity of $\mathfrak{T}(o, e, u, v)$ is equivalent to a common special case of $(v, u\,v)$- and $(u, o\,v)$-Desargues, as defined in the context of **16** (cf. Pickert 1955, p. 98). Thus if a projective plane **P** is (v, uv)- or (u, L)-transitive, for two distinct points u, v and a line $L \neq uv$ through v, then $\mathfrak{T}(o, e, u, v)$ is linear for any choice of $o \mathrel{I} L$ and e.

More than this can be proved. In fact, each of the possible Lenz-Barlotti types for projective planes corresponds to a system of algebraic laws which must be satisfied by certain well-defined ternary fields of any plane of that type. We treat only the more important cases here; other situations will occur, for the finite case, in Section 4.3 below.

A linear ternary field is called a *cartesian group*[1] if its additive loop is associative and thus a group. Note that in a cartesian group the mappings

$$(28) \qquad x \to -x a + x b \quad \textit{and} \quad x \to a x - b x$$

must be permutations whenever $a \neq b$. A *quasifield*[1] is a cartesian group $\mathfrak{T}$ satisfying

$$(29) \qquad (x + y) z = x z + y z \qquad \textit{for all } x, y, z \in \mathfrak{T};$$

it is not difficult to prove (see Pickert 1955, p. 91) that addition in any quasifield is commutative. A *semifield*[1] is a quasifield satisfying also

$$(30) \qquad x(y + z) = x y + x z \qquad \textit{for all } x, y, z \in \mathfrak{T},$$

and a *planar nearfield*[1] is a quasifield whose multiplicative loop is associative and hence a group. An *alternative field* is a semifield satisfying

$$(31) \qquad x^2 y = x(x y) \quad \textit{and} \quad x y^2 = (x y) y \qquad \textit{for all } x, y \in \mathfrak{T},$$

---

[1] The terminologies vary widely here. Cartesian groups (Pickert 1952, 1955) are called "cartesian number systems" by Baer 1942. Instead of "quasifield" (Pickert 1955), the terms "left Veblen-Wedderburn system" and "right Veblen-Wedderburn system" are customary, after Veblen & Wedderburn 1907, the choice of "left" or "right" depending on whether (29) is called the left or the right distributive law. Similarly, (32) is sometimes called the "right inverse property". Instead of "semifield" (Knuth 1965), Pickert 1955 uses "distributive quasifield", and many other authors say "division algebra" or "division ring". Planar nearfields, as defined here, are of course nearfields in the sense of Section 1.4, but not every nearfield is planar (cf. Zemmer 1964), because (28) may not be satisfied. However, it can be shown that every nearfield of finite rank over its kernel (cf. p. 132 below) is planar.

where x^2 is defined to be xx. Finally, we say that $\mathfrak{T}$ (more precisely, the multiplicative loop of $\mathfrak{T}$) has the *left inversive property*[1]) if

(32) $xx' = 1$ *implies* $x(x'y) = y$, *for all* $x, y \in \mathfrak{T}$.

Note that (32) implies that the right inverse x' of x is also a left inverse. Thus right and left inverses coincide if $\mathfrak{T}$ has the left inverse property; without (32) this need not be so

22. *Let* o, e, u, v *be a quadrangle in a projective plane* **P**, *and let* $\mathfrak{T}$ *be the ternary field* $\mathfrak{T}(o, e, u, v)$. *Also, let* Γ *denote the full collineation group* Aut**P** *of* **P**. *Then:*

(a) **P** *is* (v, uv)*-transitive if and only if* $\mathfrak{T}$ *is a cartesian group. In this case,* $\Gamma(v, uv)$ *is isomorphic to* $\mathfrak{T}^+$.
(b) **P** *is* (u, ov)*-transitive if and only if* $\mathfrak{T}$ *is linear with associative multiplication. In this case,* $\Gamma(u, ov)$ *is isomorphic to* $\mathfrak{T}^\times$.
(c) **P** *is* (uv, uv)*-transitive if and only if* $\mathfrak{T}$ *is a quasifield.*
(d) **P** *is* (v, v)*-transitive if and only if* $\mathfrak{T}$ *is a cartesian group satisfying* (30).
(e) **P** *is* (u, v)*-transitive if and only if* $\mathfrak{T}$ *is a planar nearfield.*
(f) **P** *is* (uv, uv)*- and* (v, v)*-transitive if and only if* $\mathfrak{T}$ *is a semifield.*
(g) **P** *is* (uv, uv)*- and* (ov, ov)*-transitive if and only if* $\mathfrak{T}$ *is a semifield with the left inversive property.*
(h) **P** *is* (L, L)*-transitive for every line* L *if and only if* $\mathfrak{T}$ *is an alternative field.*
(i) **P** *is desarguesian if and only if* $\mathfrak{T}$ *is a (not necessarily commutative) field.*

For the proofs of these results, see Pickert 1955, Sections 3.5 and 6.1. Some of them are easy consequences of others; for example, **(e)** and **(f)** follow immediately from **(b)**, **(c)**, **(d)**. Result **(e)** provides a proof of **19** above. The left inverse property (32), for a semifield, implies the identity

(33) $[x(yx)]z = x[y(xz)]$ *for all* $x, y, z \in \mathfrak{T}$

(Moufang 1935; Pickert 1955, p. 160; Hall 1959, p. 370); and it can be shown[2]) that (33) implies (31). Thus it follows that there exists no plane of any one of the Lenz-Barlotti types VI of **20**, as claimed in Table 1. Also, a theorem of E. Artin (Zorn 1931; Pickert 1955,

---

[1]) See footnote [1]) on p. 129.

[2]) That the first equation (31) follows from (33) is obvious: put $y = 1$. But the second equation is difficult to derive. If the semifield in question has characteristic $\neq 2$, then the second equation (31) follows from the first (Skornyakov 1951, Kleinfeld 1953). If the characteristic is 2, this is no longer true; an example (due to R. H. Bruck) is given by San Soucie 1955. But in the same paper it is shown that (33) does imply the second equation (31), also in case of characteristic 2.

p. 161—162; HALL 1959, p. 376—382) shows that a finite alternative field is associative and hence a field; this proves the claim in Table 1 that there exist no finite planes of Lenz-Barlotti type VII.1.

Let **A** be an affine plane and **P** the corresponding projective plane: $\mathbf{A} = \mathbf{P}^W$ for some line W of **P**. A collineation of **A** is called a *dilatation* if it has axis W when regarded as a collineation of **P**. A *translation* of **A** is a fixed point free dilatation or the identity (compare here the terminology of Section 2.3), i.e. an elation of **P** with axis W. We call **A** a *translation plane* if the group of all translations of **A** is transitive on the points of **A**, in other words if **P** is (W, W)-transitive. By **22c**, the translation planes are precisely those affine planes which can be coordinatized by a quasifield. Translation planes have been well investigated, and almost all known finite planes are either translation planes or closely related to them, as will be seen in Chapter 5. For this reason we devote the remainder of this section to some general results on translation planes and related concepts.

Let **A** be a translation plane and T its (full) translation group. Again we put $\mathbf{A} = \mathbf{P}^W$ and consider T also as the group of all elations with axis W of the projective plane **P**. By (14), we have $\mathrm{T} = \mathrm{T}(W, W) = \bigcup_{x \mathrm{I} W} \mathrm{T}(x, W)$, and as the $x \mathrm{I} W$ are just the parallel classes of **A**, we can write

$$\mathrm{T} = \bigcup_{\mathfrak{X}} \mathrm{T}(\mathfrak{X}), \tag{34}$$

where $\mathfrak{X}$ ranges over the parallel classes of **A** and $\mathrm{T}(\mathfrak{X})$ denotes the subgroup of T which fixes every line of $\mathfrak{X}$. It follows from **4** that

$$\mathrm{T}(\mathfrak{X}) \cap \mathrm{T}(\mathfrak{Y}) = 1 \qquad \textit{if } \mathfrak{X} \neq \mathfrak{Y}, \tag{35}$$

so that the $\mathrm{T}(\mathfrak{X})$ form a *partition* of T, as defined in Section 1.2. Furthermore, this is even a *congruence partition* of T, in the sense that

$$\textit{if} \quad \mathfrak{X} \neq \mathfrak{Y}, \quad \textit{then} \quad \mathrm{T}(\mathfrak{X})\,\mathrm{T}(\mathfrak{Y}) = \mathrm{T}.^{1)} \tag{36}$$

Conversely, if T is an abstract group possessing a nontrivial congruence partition, i.e. a set $\mathscr{C}$ of proper subgroups $\mathrm{T}(\mathfrak{X})$ satisfying (34)—(36) [here $\mathfrak{X}$ ranges over some index set of cardinality $\geqq 2$], then T may be regarded as the full translation group of a translation plane. In fact, the incidence structure $\mathbf{J} = \mathbf{J}(\mathrm{T}, \mathscr{C})$ of Section 1.2, whose points are the elements of T and whose blocks are the cosets²) of the $\mathrm{T}(\mathfrak{X})$, is a translation plane and T its full translation group. Hence:

---

¹) This implies, of course, that $\mathrm{T}(\mathfrak{X})$ and $\mathrm{T}(\mathfrak{Y})$ generate T. There are examples showing that (36) cannot be replaced by this weaker property if the following converse is to hold.

²) Left and right cosets are identical here, for (34)—(36) imply that T is abelian (ANDRÉ 1954a, Satz 7).

23. *There is a canonical correspondence*[1]) *between translation planes and congruence partitions.*

Proof: ANDRÉ 1954a, Satz 9.

The *kernel of the translation plane* **A** is the set $\mathfrak{K}(\mathbf{A})$ of all endomorphisms α of T with

$$(37) \qquad \mathsf{T}(\mathfrak{X})^{\alpha} \subseteqq \mathsf{T}(\mathfrak{X}), \qquad \textit{for all parallel classes } \mathfrak{X}.$$

With the usual addition and multiplication of endomorphisms, $\mathfrak{K}(\mathbf{A})$ is a ring, and it can be shown (ANDRÉ 1954a, Satz 10) that $\mathfrak{K}(\mathbf{A})$ is even a field[2]). The *kernel of a quasifield* $\mathfrak{Q}$ is the set $\mathfrak{K}(\mathfrak{Q})$ of all $k \in \mathfrak{Q}$ with

$$(38) \quad k(xy) = (kx)\,y \quad \textit{and} \quad k(x+y) = k\,x + k\,y, \qquad \textit{for all } x, y \in \mathfrak{Q}.$$

Clearly, $\mathfrak{K}(\mathfrak{Q})$ is also a field, and in fact

24. *If* $\mathfrak{Q}$ *is any*[3]) *coordinatizing quasifield of the translation plane* **A**, *then* $\mathfrak{K}(\mathfrak{Q}) \cong \mathfrak{K}(\mathbf{A})$.

Proof: ANDRÉ 1954a, p. 174—176. This result justifies the choice of the same term "kernel" for two seemingly unrelated concepts.

By (38), a quasifield $\mathfrak{Q}$ may be regarded as a left vector space over any subfield $\mathfrak{F}$ of its kernel $\mathfrak{K}(\mathfrak{Q})$. Also, the group T may be regarded as the additive group of a vector space over $\mathfrak{F}$, and if $[\mathfrak{Q}:\mathfrak{F}]$ and $[\mathsf{T}:\mathfrak{F}]$ denote the respective ranks of these vector spaces, then

$$(39) \qquad [\mathsf{T}:\mathfrak{F}] = 2[\mathfrak{Q}:\mathfrak{F}]$$

(ANDRÉ 1954a, p. 181). Thus if one of these ranks is finite, then they both are, and $[\mathsf{T}:\mathfrak{F}]$ is even. Clearly, $\mathfrak{K}(\mathfrak{Q}) = \mathfrak{Q}$ if and only if $\mathfrak{Q}$ is a field; hence **22i** shows that

25. **A** *is desarguesian if and only if* $[\mathsf{T}:\mathfrak{K}(\mathbf{A})] = 2$.

The subgroups $\mathsf{T}(\mathfrak{X})$ of the congruence partition associated with the translation plane **A** are, by (37), subspaces of the vector space T over $\mathfrak{F} \subseteqq \mathfrak{K}(\mathbf{A})$. Hence in the corresponding desarguesian projective

---

[1]) In **23** and **26**, the term "canonical correspondence" is to be understood in the following sense: The isomorphism classes of translation planes [in **26** with condition (**a**)] can be put into a one-one correspondence with the isomorphism classes of groups with congruence partitions [of projective $(2t+1)$-spaces over $\mathfrak{K}$ with t-spreads].

[2]) As always in this book, "fields" need not be commutative. The field $\mathfrak{K}(\mathbf{A})$ was first considered in a more special situation by ARTIN 1940.

[3]) Two such quasifields are usually not isomorphic; cf. **21**. In general, it is a difficult question to decide whether or not two quasifields with isomorphic kernels coordinatize the same translation plane. Complicated necessary and sufficient conditions for this were given by SKORNYAKOV 1949. Some of the questions involved will be discussed in a more special situation further below; cf. **32** and **34**.

geometry $\mathscr{P} = \mathscr{P}(\mathrm{T})$ of the vector space T [cf. **1.4.2**], these subgroups define a *spread*, i.e. a collection $\mathscr{S}$ of mutually disjoint subspaces covering all of $\mathscr{P}$ [cf. **1.4.6**]; and $\mathscr{S}$ has the further property that any two distinct subspaces of $\mathscr{S}$ span $\mathscr{P}$. Conversely, every spread of $\mathscr{P}$ with this property defines a congruence partition of the additive group of the vector space underlying $\mathscr{P}$, and hence a translation plane, by **23**. In the finite-dimensional case, the situation is as follows (terminology as in Section 1.4):

26. *Let* $\mathfrak{F}$ *be a field and* t *a positive integer. Then there is a canonical correspondence*[1]) *between*

(a) *the translation planes* **A** *with* $\mathfrak{F} \subseteqq \mathfrak{K}(\mathbf{A})$ *and* $[\mathfrak{Q}:\mathfrak{F}] = t + 1$, *for an arbitrary coordinatizing quasifield* $\mathfrak{Q}$ *of* **A**, *and*

(b) *the t-spreads of the* $(2t + 1)$*-dimensional projective geometry over* $\mathfrak{F}$.

Proof: André 1954a, p. 182, where it is also pointed out that different choices of $\mathfrak{F} \subseteqq \mathfrak{K}(\mathbf{A}) \cong \mathfrak{K}(\mathfrak{Q})$ yield different spread representations of **A**. Result **26** was rediscovered independently by Bruck & Bose 1964 and Segre 1964.

The following results are concerned with representations of certain types of collineations in a translation plane **A** with coordinatizing quasifield $\mathfrak{Q}$.

27. *The translations of* **A** *are the mappings*

$$(40)\qquad \tau(s,t):\ (x,y) \to (x+s,\,y+t),\qquad s,t \in \mathfrak{Q};$$

and the dilatations with center $(0,0)$ *are the mappings*

$$(41)\qquad \delta(k):\ (x,y) \to (kx,\,ky),\qquad k \in \mathfrak{K}(\mathfrak{Q}).$$

The proof [André 1954a, (7) and (9)] is straightforward. As a consequence of (41), we note:

28. *For any point* p *of a translation plane* **A**, *the group of dilatations with center* p *is isomorphic to the multiplicative group of the kernel* $\mathfrak{K}(\mathbf{A})$.

We consider now axial collineations in **A**. Such a collineation is central in the corresponding projective plane **P** with $\mathbf{A} = \mathbf{P}^W$, and since W must stay fixed, the center is on W. Thus:

29. *A nontrivial collineation* φ *of an affine plane* **A** *which has an axis in* **A** *has no center in* **A**. *Instead,* φ *fixes every line of a unique parallel class.*

If the axis belongs to this parallel class, we call φ a *shear*, otherwise a *strain*. We give now representations of shears with axis *ov* (hence

---

[1]) See footnote [1]) on p. 132.

with center v) and of strains with axes ov and ou (and corresponding centers u, v), in the translation plane **A** over the quasifield $\mathfrak{Q}$. The *distributor* of $\mathfrak{Q}$ is the set $\mathfrak{D} = \mathfrak{D}(\mathfrak{Q})$ of all $d \in \mathfrak{Q}$ with

$$(42) \qquad x(d + y) = x\,d + x\,y \qquad \textit{for all } x, y \in \mathfrak{Q}.$$

The *middle* and *right nucleus*[1] of $\mathfrak{Q}$ are the sets $\mathfrak{M}(\mathfrak{Q})$ and $\mathfrak{R}(\mathfrak{Q})$ of those $m \in \mathfrak{Q}$ and $r \in \mathfrak{Q}$ for which

$$(43) \qquad (x\,m)\,y = x(m\,y)$$

or, respectively,

$$(44) \qquad (x\,y)\,r = x(y\,r)$$

holds, for all $x, y \in \mathfrak{Q}$. We then have the following results.

30. *The shears with axis ov, of the translation plane* **A** *over the quasifield $\mathfrak{Q}$, are the mappings*

$$(45) \qquad \sigma(d)\colon \quad (x, y) \to (x, xd + y), \qquad d \in \mathfrak{D}(\mathfrak{Q}).$$

The strains with center u and axis ov are the mappings

$$(46) \qquad \mu(m)\colon \quad (x, y) \to (xm, y), \qquad m \in \mathfrak{M}(\mathfrak{Q}),$$

and the strains with center v and axis ou are the mappings

$$(47) \qquad \varrho(r)\colon \quad (x, y) \to (x, yr), \qquad r \in \mathfrak{R}(\mathfrak{Q}).$$

The proofs are again straightforward; see ANDRÉ 1955, (4), (7), (7′). These results generalize parts of **22**: the projective plane **P** with $\mathbf{A} = \mathbf{P}^W$ is (uv, uv)-transitive by hypothesis, and it is (v, v)-transitive it and only if $\mathfrak{D}(\mathfrak{Q}) = \mathfrak{Q}$, i.e. if $\mathfrak{Q}$ is a semifield (cf. **22c**, **d**). Also, **P** is (u, ov)-transitive if and only if $\mathfrak{M}(\mathfrak{Q}) = \mathfrak{R}(\mathfrak{Q}) = \mathfrak{Q}$, in which case it must then also be (v, ou)-transitive (cf. **19** and **22b**); $\mathfrak{Q}$ is then clearly a planar nearfield.

We give a few more details for those special cases where $\mathfrak{Q}$ is a semifield or a planar nearfield. First, it is straightforward to prove that

31. *The nuclei $\mathfrak{M}(\mathfrak{Q})$ and $\mathfrak{R}(\mathfrak{Q})$ of a semifield $\mathfrak{Q}$ are fields.*

Thus **30** shows that the groups of (u, ov)- and (v, ou)-strains are isomorphic to the multiplicative groups of these fields. If $\mathfrak{F}$ is the intersection of the fields $\mathfrak{K}(\mathfrak{Q})$, $\mathfrak{M}(\mathfrak{Q})$ and $\mathfrak{R}(\mathfrak{Q})$, then $\mathfrak{Q}$ may be regarded as a left vector space over $\mathfrak{F}$. Suppose that $\mathfrak{Q}$ and $\mathfrak{S}$ are two semifields with the same $\mathfrak{F}$. Then by an *isotopism* from $\mathfrak{Q}$ to $\mathfrak{S}$ is meant

---

[1] The *left nucleus* $\mathfrak{L}(\mathfrak{Q}) = \{l \in \mathfrak{Q} : (l\,x)\,y = l(x\,y) \text{ for all } x, y \in \mathfrak{Q}\}$ bears less significance in the present context than $\mathfrak{M}(\mathfrak{Q})$ and $\mathfrak{R}(\mathfrak{Q})$. Note that $\mathfrak{L}(\mathfrak{Q})$ contains $\mathfrak{K}(\mathfrak{Q})$, and in fact $\mathfrak{L}(\mathfrak{Q}) = \mathfrak{K}(\mathfrak{Q})$ if and only if $\mathfrak{Q}$ is a semifield.

a triple (α, β, γ) of nonsingular linear transformations from $\mathfrak{Q}$ onto $\mathfrak{S}$ (both considered as vector spaces over $\mathfrak{F}$) such that

$$x^{\alpha} \cdot y^{\beta} = (x\,y)^{\gamma} \qquad \textit{for all } x, y \in \mathfrak{Q}.$$

Here multiplication is written $x\,y$ in $\mathfrak{Q}$ and $x \cdot y$ in $\mathfrak{S}$. If there exists an isotopism from $\mathfrak{Q}$ onto $\mathfrak{S}$, then $\mathfrak{Q}$ and $\mathfrak{S}$ are said to be *isotopic.*

32. *Two semifields coordinatize isomorphic translation planes if and only if they are isotopic.*[1]

Proof: ALBERT 1960, Section 9. An isotopism of $\mathfrak{Q}$ onto itself is called an *autotopism.* If φ is a collineation which fixes the points $o = (0, 0)$, $u = (0)$ and v of the translation plane **A** over the semifield $\mathfrak{Q}$, then $(x, 0)\varphi = (x^{\alpha}, 0)$, $(a)\varphi = (a^{\beta})$ [cf. (22)], and $(0, y)\varphi = (0, y^{\gamma})$, for three well-defined permutations α, β, γ of $\mathfrak{Q}$. It follows easily that (α, β, γ) is an autotopism of $\mathfrak{Q}$, and in fact one may prove now:

33. *The autotopisms of a semifield $\mathfrak{Q}$ form a group isomorphic to the stabilizer of the points o, u, v, in the full collineation group of the translation plane over $\mathfrak{Q}$.*

This stabilizer has a complement in the group Γ of all collineations fixing v and uv. (Unless $\mathfrak{Q}$ is a field, Γ is the full collineation group, because the plane must then be of Lenz-Barlotti type V.1.) This complement is the subgroup Σ generated by all translations (40) and shears (45). For an elegant representation of the plane within this metabelian group Σ, see CRONHEIM 1965.

Now let **A** be a translation plane over a planar nearfield $\mathfrak{Q}$ which is not a field. Then **A** may be coordinatized also by quasifields not isomorphic to $\mathfrak{Q}$. No such quasifield can be also a nearfield:

34. *If $\mathfrak{Q}$ and $\mathfrak{Q}'$ are planar nearfields coordinatizing the same translation plane, then $\mathfrak{Q}$ and $\mathfrak{Q}'$ are isomorphic.*

Proof: ANDRÉ 1955, Satz 7. A nondesarguesian translation plane can be coordinatized by a planar nearfield if and only if it is of Lenz-Barlotti type IVa.2 or IVa.3; this shows that any collineation fixing v must also fix u, so that in particular there are no shears with axis ov in such a plane. In view of **30**, we can conclude:

35. *The distributor $\mathfrak{D}(\mathfrak{Q})$ of any planar nearfield $\mathfrak{Q}$ which is not a field consists of* 0 *only.*

Further results on planes over planar nearfields, particularly their collineation groups, are found in ANDRÉ 1955. Some of these results (for the finite case) will be given in Section 5.2.

---

[1] Compare here footnote [3]) on p. 132. There is a generalization of **34** to arbitrary ternary fields; see KNUTH 1965, Theorem 3.3.2; cf. also SANDLER 1964.

We have remarked above that a translation plane is essentially the same as a (W, W)-transitive projective plane. We shall now make a few remarks about the dual concept. Let $\mathbf{P}$ be a projective plane which is (v, v)-transitive for some point v. Let W be a line through v; then the affine plane $\mathbf{A} = \mathbf{P}^W$ is a *shears plane* in the sense that

(48) *There is a parallel class* $\mathfrak{B}$ *of lines* (*with ideal point* v) *such that, for every* $X \in \mathfrak{B}$, *the group of shears with axis* X *is transitive on the ideal points* $\neq v$.

It follows from **22d** that every shears plane can be coordinatized by a cartesian group satisfying (30). If $\mathfrak{Q}$ is an arbitrary quasifield, then the following system $\mathfrak{Q}^*$ is clearly such a cartesian group: Addition is the same in $\mathfrak{Q}$ and $\mathfrak{Q}^*$, and the multiplication $*$ in $\mathfrak{Q}^*$ is related to that in $\mathfrak{Q}$ by

(49) $$x * y = y x \qquad \text{for all } x, y \in \mathfrak{Q}.$$

Also, it is clear that every cartesian group satisfying (30) must be of this form $\mathfrak{Q}$. Moreover:

36. *For any quasifield* $\mathfrak{Q}$, *the projective planes coordinatized by* $\mathfrak{Q}$ *and* $\mathfrak{Q}^*$ *are dual to each other.*

For the proof of this, see Pickert 1955, pp. 41, 91.

Finally, we discuss a class of affine planes closely related to translation planes. We say that the affine plane $\mathbf{A}$ is a *semi-translation plane*[1]) if it contains a Baer subplane $\mathbf{B}$ [in the affine sense, as defined by (13)], such that

(50) $\mathbf{B}$ *is a translation plane,*

and

(51) *Every translation of* $\mathbf{B}$ *is induced by a translation of* $\mathbf{A}$.

Given a semi-translation plane $\mathbf{A} = \mathbf{P}^W$ with Baer subplane $\mathbf{B} = \mathbf{Q}^W$, select o, e, u, v in $\mathbf{Q}$, with $u, v \mathrm{I} W$, and consider the ternary field $\mathfrak{T} = \mathfrak{T}(o, e, u, v)$ of $\mathbf{P}$. It is clear that $\mathfrak{T}$ will contain a ternary subfield $\mathfrak{Q}$ coordinatizing $\mathbf{Q}$, and $\mathfrak{Q}$ is a quasifield, because of (50). Also, (51) implies that the translations of $\mathbf{B}$ extend to translations

(52) $$(x, y) \to (x + s, y + t) \qquad x, y \in \mathfrak{T};\ s, t \in \mathfrak{Q}$$

---

[1]) Ostrom 1964a uses this term in a slightly different sense; his definition applies only to finite planes, but even then ours is more special. Pickert 1965a calls the planes considered here "normal" semi-translation planes. Note that the term is somewhat unfortunate insofar as a translation plane is not necessarily a semi-translation plane. For example, the desarguesian affine planes $\mathbf{A}(q)$, for $q = p^e$ with odd e, do not contain Baer subplanes and are therefore not semi-translation planes.

of **A**; here the addition is that of $\mathfrak{T}$, as defined by (26). The fact that (52) describes a collineation yields the following properties of $\mathfrak{T}$:

$$(53)\qquad \left.\begin{array}{c} (x+s)+y = x+(s+y) \\ (x+y)+s = x+(y+s) \\ (x+y)\,t = x\,t + y\,t \end{array}\right\} \text{for } x, y \in \mathfrak{T};\ s, t \in \mathfrak{Q};$$

here multiplication is that defined by (26). Also, it follows that

(54) *if* $z \in \mathfrak{T} - \mathfrak{Q}$, *then* $(s, t) \to s\,z + t$ *is one-one from* $\mathfrak{Q} \times \mathfrak{Q}$ *onto* $\mathfrak{T}$.

Hence much of the structure of $\mathfrak{T}$ is determined by addition and multiplication. $\mathfrak{T}$ will be termed *$\mathfrak{Q}$-linear*[1]) provided that

$$(55)\qquad x \cdot s \circ y = x\,s + y \qquad \text{for } x, y \in \mathfrak{T} \text{ and } s \in \mathfrak{Q}.$$

Now the following converse holds:

37. *Let $\mathfrak{T}$ be an algebraic system with two binary operations, addition and multiplication, and suppose that $\mathfrak{T}$ contains a subsystem $\mathfrak{Q}$ which is a quasifield with respect to these operations, such that* (53) *and* (54) *are satisfied. Define*

$$(56)\qquad x \cdot y \circ z = \begin{cases} x\,y + z & \text{if } y \in \mathfrak{Q} \\ (x+s)\,y + t & \text{if } y \notin \mathfrak{Q} \text{ and } z = s\,y + t; \quad s, t \in \mathfrak{Q}. \end{cases}$$

If this turns $\mathfrak{T}$ into a ternary field, then $\mathfrak{T}$ is $\mathfrak{Q}$-linear and coordinatizes a semi-translation plane **A**, *with Baer subplane* **B** *coordinatized by* $\mathfrak{Q}$.

For the proof see Ostrom 1964a; Pickert 1965a, Satz 2. Further results in this direction are found in Morgan & Ostrom 1964. Result **37** will be used in Section 5.4 for the construction of finite semi-translation planes which are not translation planes.

3.2 Combinatorics of finite planes

The definitions in Sections 2.1, 2.2, and 3.1 imply immediately that the finite projective [affine] planes are precisely the projective [affine[2])] designs with $\lambda = 1$. We shall present some combinatorial properties of these designs in this section. The blocks will always be referred to as lines; this is consistent with the definition of lines in arbitrary designs, as given in Section 2.1.

By definition (2.1.9), the *order* of a finite projective or affine plane is the integer n determined by the condition that the number of lines

---

[1]) Whether or not $\mathfrak{T}$ must be automatically $\mathfrak{Q}$-linear for every semi-translation plane seems to be an open question. (It appears rather likely that this need not be the case.)

[2]) In the affine case it suffices to demand that the design have a parallelism; cf. (2.2.5). That we must then have an affine design, and hence an affine plane, follows from result **2.2.6**.

through any point is $n+1$. Note that if $\mathbf{P}$ is a finite projective plane and $\mathbf{A} = \mathbf{P}^W$ for some line W of $\mathbf{P}$, then $\mathbf{P}$ and $\mathbf{A}$ have the same order. The following facts are easily verified:

1. *If* $\mathbf{P} = (\mathfrak{p}, \mathfrak{L}, \mathrm{I})$ *is a projective plane of order* n, *then*

$$|\mathfrak{p}| = n^2 + n + 1, \tag{1}$$

$$|\mathfrak{L}| = n^2 + n + 1, \tag{2}$$

$$[p] = n + 1, \quad \text{for every } p \in \mathfrak{p}, \tag{3}$$

$$[L] = n + 1, \quad \text{for every } L \in \mathfrak{L}, \tag{4}$$

$$[p, q] = 1, \quad \text{for } p, q \in \mathfrak{p} \quad \text{and} \quad p \neq q; \tag{5}$$

$$[L, M] = 1, \quad \text{for } L, M \in \mathfrak{L} \quad \text{and} \quad L \neq M. \tag{6}$$

2. *If* $\mathbf{A} = (\mathfrak{p}, \mathfrak{L}, \mathrm{I})$ *is an affine plane of order* n, *then* (3) *and* (5) *hold, and*

$$|\mathfrak{p}| = n^2, \tag{7}$$

$$|\mathfrak{L}| = n(n+1), \tag{8}$$

$$[L] = n, \quad \text{for every } L \in \mathfrak{L}. \tag{9}$$

The converses of **1** and **2** are likewise true; in fact (1)—(6) and (3), (5), (7)—(9) are then redundant sets of conditions, as will be seen in **3** and **4** below. Before stating these results, we make the following convention: For $1 \leqq i \leqq 9$, let (i') and (i'') stand for condition (i), with "$=$" replaced by "$\leqq$" or "$\geqq$", respectively.

3. *Each of the following conditions on the integer* $n > 1$ *is necessary and sufficient for the nondegenerate incidence structure* $(\mathfrak{p}, \mathfrak{L}, \mathrm{I})$ *to be a projective plane of order* n:

(a)	(3′), (4), (5),	*or dually*	(3), (4′), (6);
(b)	(4), (5), (6),	*or dually*	(3), (5), (6);
(c)	(1), (4), (5),	*or dually*	(2), (3), (6);
(d)	(2), (4″), (5),	*or dually*	(1), (3″), (6);
(e)	(1), (4), (5″), (6′),	*or dually*	(2), (3), (5′), (6″);
(f)	(1″), (4′), (5), (6″),	*or dually*	(2″), (3′), (5″), (6);
(g)	(1″), (4′), (5″), (6),	*or dually*	(2″), (3′), (5), (6″);
(h)	(2), (4″), (5″), (6′),	*or dually*	(1), (3″), (5′), (6″);
(i)	(2″), (4), (5), (6″),	*or dually*	(1″), (3), (5″), (6);
(j)	(2″), (4), (5″), (6),	*or dually*	(1″), (3), (5), (6″);
(k)	(1′), (2″), (4″), (5′),	*or dually*	(1″), (2′), (3″), (6′);
(l)	(1″), (2′), (4′), (5″),	*or dually*	(1′), (2″), (3′), (6″);
(m)	(1″), (2′), (3″), (5′),	*or dually*	(1′), (2″), (4″), (6′);
(n)	(2″), (3′), (4), (5″),	*or dually*	(1″), (3), (4′), (6″);
(o)	(2), (3″), (4″), (5′),	*or dually*	(1), (3″), (4″), (6′);

(p) (2″), (3′), (4″), (5), *or dually* (1″), (3″), (4′), (6);
(q) (2″), (3), (4′), (6″), *or dually* (1″), (3′), (4), (5″);
(r) (2), (3″), (4″), (6′), *or dually* (1), (3″), (4″), (5′);
(s) (2″), (3″), (4), (6), *or dually* (1″), (3), (4″), (5);
(t) (1″), (3″), (4′), (5′), (6″), *or dually* (2″), (3′), (4″), (5″), (6′);
(u) (1″), (3′), (4′), (5″), (6″), *or dually* (2″), (3′), (4′), (5″), (6″);
(v) (1″), (3′), (4″), (5″), (6′), *or dually* (2″), (3″), (4′), (5′), (6″);
(w) (1′), (2″), (3′), (4″), (5″), *or dually* (1″), (2′), (3″), (4′), (6″).

For the proof, see Corsi 1963; parts of **3** are also in Hall 1959, p. 392 and Barlotti 1962. Conditions **(c)**—**(w)** constitute a complete list of those systems taken from (1)—(6″) which (i) axiomatize finite projective planes and (ii) cannot be weakened by the omission of further conditions (1)—(6″) without losing property (i).[1]

A similarly thorough investigation for finite affine planes does not seem to exist. We note here the following analogue:

4. *Each of the following conditions on the integer $n > 1$ is necessary and sufficient for the nondegenerate incidence structure* $(\mathfrak{p}, \mathfrak{L}, \mathrm{I})$ *to be an affine plane of order n:*

(a) (3), (5), (9);
(b) (5), (7), (9);
(c) (5′), (8), (9);
(d) (3′), (5″), (7″), (9′);
(e) (3″), (5′), (7′), (9″);
(f) (3′), (5″), (8″), (9);
(g) (3″), (5′), (8′), (9);
(h) (5′), (7″), (8′), (9);
(i) (5″), (7′), (8″), (9).

The proofs consist in simple generalizations of arguments in Ostrom 1964, Lemma 8, and Dembowski & Ostrom 1968, Lemma 11.

Next, we give two characterizations of finite projective planes by certain sets of permutations. The first of these is the special case $\lambda = 1$ of **2.1.18**:

5. *A projective plane of order n exists if and only if there exists a set Σ of permutations of a set $\mathfrak{p}$ with $|\mathfrak{p}| = n^2 + n + 1$, satisfying*

(a) *$x^\varrho = x^\sigma$ for some $x \in \mathfrak{p}$ and $\varrho, \sigma \in \Sigma$ implies $\varrho = \sigma$, and*

---

[1] In Corsi's paper, conditions **(a)** and **(b)** are not considered because they are also satisfied in the degenerate case $\mathfrak{p} = \mathfrak{L} = \emptyset$ and hence cannot serve as axioms for finite projective planes. This is the reason for including the word "nondegenerate" in the hypothesis of **3**.

(**b**) *If $x, y \in \mathfrak{p}$ and $x \neq y$, then there is a unique pair ϱ, σ of permutations in Σ such that $x^{\varrho\sigma^{-1}} = y$.*

Note that (**a**) and (**b**) imply that Σ cannot be a group, and $|\Sigma| = n + 1$.

The second characterization involves permutations of degree n rather than $n^2 + n + 1$:

6. *A projective plane of order n exists if and only if there exists a sharply 2-transitive set of permutations of degree n.*

Note that this set need not be a group; it must consist of $n(n-1)$ permutations. Result **6** can be found in Section IV of WITT 1938b; see also HALL 1943, Theorem 5.2, and Appendix I. For the proof, label the lines $\neq uv$ through u and the lines $\neq uv$ through v (where u, v are distinct points of a plane $\mathbf{P}$ of order n) by the integers $1, \ldots, n$, and call (i, j) the intersection point of line i through v and line j through u. Then any line $L \,\mathrm{I}\, u, v$ of the affine plane $\mathbf{P}^{uv}$ consists of the points (i, i^{π}), $i = 1, \ldots, n$, where $\pi = \pi(L)$ is a permutation of $\{1, \ldots, n\}$, well defined by L. The set Π of all these $\pi(L)$ is sharply 2-transitive on $\{1, \ldots, n\}$, and conversely, every such sharply 2-transitive set can be interpreted in this fashion. If coordinates are introduced in $\mathbf{P}$ with u, v as in Section 3.1, then the permutations $\pi(L)$ are essentially the same as the mappings

$$x \to x \cdot a \circ b, \quad \text{with} \quad a \neq 0, \tag{10}$$

of the ternary field $\mathfrak{T}(o, e, u, v)$ onto itself, for any admissible choice of o and e. In fact, if L has equation $y = x \cdot a \circ b$, then (10) is $\pi(L)$.

Contrary to the situation in **5**, a sharply 2-transitive set Π of permutations of degree n may well be a group. In fact, if this is the case, Π may be identified with the set of permutations

$$x \to x a + b, \quad \text{with} \quad a \neq 0, \tag{11}$$

of a finite nearfield.[1] It can be shown (HALL 1943, Theorem 5.7) that this nearfield is essentially identical with $\mathfrak{T}(o, e, u, v)$. Thus:

7. *A sharply 2-transitive set Π of permutations of degree n is a group if and only if the ternary field given by* (10) *is a nearfield; in this case the plane determined by Π is of Lenz-Barlotti type* IVa.2, IVa.3, *or* VII.2.

We shall now discuss several types of substructures of finite projective and affine planes; these will be important either for the problem of constructing such planes or for dualities or collineations of them.

---

[1] This seems to have been first proved by CARMICHAEL 1931b; see also CARMICHAEL 1937, Chapter 13; ZASSENHAUS 1935a, b; and HALL 1943.

A *net*[1] is a nondegenerate partial plane satisfying the parallel axiom (3.1.4); in other words, a net is an incidence structure of points and lines such that

(12) *There exist points and lines, and to every point (line) there exist two lines (points) not incident with it.*
(13) $[p, q] \leqq 1$ *for any two distinct points* p, q.
(14) *If* $p \not\mathrm{I}\, L$, *then there exists one and only one line* $M \mathrm{I}\, p$ *such that* $[M, L] = 0$.

Condition (14) permits the introduction of a *parallelism*[2] in any net:

$$L \parallel M \quad \text{if and only if} \quad L = M \quad \text{or} \quad [L, M] = 0.$$

This is an equivalence relation among the lines of the net, with the following property:

(15) *Every point is on exactly one line of each parallel class.*

If $\mathbf{A}$ is an affine plane and $\mathfrak{U}$ a union of complete parallel classes of lines in $\mathbf{A}$, then the points of $\mathbf{A}$ and the lines of $\mathfrak{U}$ obviously form a net. In particular, $\mathbf{A}$ is itself a net. Conversely, the question arises under what circumstances a given net may be interpreted in this fashion as a union $\mathfrak{U}$ of parallel classes in an affine plane. A net which can be described in this way will be called *imbeddable*; we shall see that there exist many non-imbeddable finite nets.

8. *Every finite net is a tactical configuration whose parameters satisfy*

$$(16) \qquad v = k^2, \quad b = r\,k, \quad r \leqq k + 1.$$

Furthermore, the number of parallel classes is r, *and every parallel class consists of* k *lines. A finite net is an affine plane if and only if* $r = k + 1$.

When speaking of finite nets, we shall henceforth often denote the parameter k by n and call it the *order* of the net. In case of imbeddability, $k = n$ is clearly the order of any imbedding affine plane. The parameter r, i.e. the number of parallel classes, is of obvious importance for the problem of imbeddability; for this reason we shall often refer to a net with parameters (16) as an *r-net.*

For the following results concerning the imbeddability problem, it is convenient to define

$$(17) \qquad f(x) = \frac{x}{2}[x^3 + 3 + 2x(x + 1)].$$

---

[1] There is an extensive literature on (not necessarily finite) nets: Blaschke & Bol 1938; Baer 1939, 1940; Pickert 1955, Kap. 2. For the topics discussed here, see Levi 1942; Bruck 1951, 1963a; Ostrom 1964c, 1965b, 1966b, 1968.

[2] Cf. (2.2.5).

We then have:

9. *Let* **N** *be an r-net of order n.*

(a) *If $n > f(n - r)$, then* **N** *is imbeddable.*

(b) *If* **N** *is imbeddable and $n > (n - r)^2$, then the imbedding affine plane is unique.*

Also, since $f(x) > x^2$ for all positive x, the imbedding is unique in case (**a**). These results are due to Bruck 1963a, Theorems 4.3 and 3.1. (See also Bose 1963b.) That (**b**) is a best possible result was established by Ostrom 1964c: he showed that if $n = (n - r)^2$, then there are at most two nonisomorphic imbeddings, and he gave examples where there are actually two.

We shall now set up a fundamental relationship between r-nets and sets of $r - 2$ mutually orthogonal Latin squares. A *Latin square* of order n is an (n, n)-matrix $L = (l_{ij})$ whose entries are the integers $1, \ldots, n$,[1] such that

(18) *Each of the integers $1, \ldots, n$ occurs exactly once in every row and every column of L.*

Two Latin squares $L = (l_{ij})$ and $L' = (l'_{ij})$ are called *orthogonal* to each other provided that

(19) *Each of the n^2 ordered pairs (s, t), where $1 \leqq s, t \leqq n$, occurs exactly once among the ordered pairs (l_{ij}, l'_{ij}), $l \leqq i, j \leqq n$.*

The correspondence between nets and Latin squares is the following. Given an r-net **N** of order n, select two parallel classes $\mathfrak{R}$ and $\mathfrak{C}$ of **N**, and number their lines in an arbitrary but fixed fashion:

$$\mathfrak{R} = \{R_1, \ldots, R_n\}, \qquad \mathfrak{C} = \{C_1, \ldots, C_n\}.$$

An arbitrary point p of **N** is then on exactly one line R_i and exactly one line C_j; hence p may be denoted by (i, j). If $\mathfrak{X}$ is any one of the remaining $r - 2$ parallel classes of **N**, number its lines

$$\mathfrak{X} = \{X_1, \ldots, X_n\}$$

and define $L = L(\mathfrak{X}) = (l_{ij})$ as follows: $l_{ij} = m$ if the unique line of $\mathfrak{X}$ through (i, j) is X_m $(1 \leqq m \leqq n)$. It is straightforward to check that each $L(\mathfrak{X})$ is a Latin square, and that these $r - 2$ Latin squares are mutually orthogonal. Conversely, any set of $r - 2$ mutually orthogonal Latin squares of order n gives rise to an r-net of order n (Bruck 1951; for the case $r = n + 1$ see Bose 1938). Thus:

10. *An r-net of order n exists if and only if a set of $r - 2$ mutually orthogonal Latin squares of order n exists. In particular, there exist (pro-*

---

[1] That $l_{ij} \in \{1, \ldots, n\}$ is not essential but convenient. It is sufficient to have any n distinct symbols as entries of L.

jective and affine) planes of order n if and only if there exist n — 1 mutually orthogonal Latin squares of order n.

In view of this result, the maximal number $N(n)$ of mutually orthogonal Latin squares of order n is of obvious importance for the existence problem of finite nets and planes. This function is one of the oldest objects of study in combinatorial mathematics.[1] We list some of the more important properties of this function.

11. *The maximal number $N(n)$ of mutually orthogonal Latin squares of order n satisfies:*

(**a**) $N(n) \leqq n - 1$;

(**b**) $N(q) = q - 1$ *if q is a prime power;*

(**c**) $N(n\,m) \geqq \min(N(n), N(m))$.

For proofs see, for example, Ryser 1963, p. 80—84. Note that, in view of **10**, result **11b** follows immediately from the existence of projective planes for every prime power order q, viz., the desarguesian planes $\mathbf{P}(q)$. Combination of **11b** and **11c** gives

$$N\left(\prod_{i=1}^{m} p_i^{e_i}\right) \geqq \min\{p_i^{e_i} - 1 : i = 1, \ldots, m\} \tag{20}$$

(MacNeish 1922). MacNeish also conjectured that equality holds in (20); that this is false was first shown by Parker 1959a by the following theorem, whose proof depends on **6**:

12. *If there exists a projective plane of order n, and if there exists a design with parameters $v, b, r, k = n, \lambda = 1$, then $N(v) \geqq n - 2$.*

For example, if n and $n + 1$ are both prime powers (i.e. $n = 8$ or n a Mersenne prime or $n + 1$ a Fermat prime), then $N(n^2 + n + 1) \geqq n - 1$, and in this particular situation Parker (1959a, Theorem 2) has shown that even $N(n^2 + n + 1) \geqq n$, while (20) would yield only $N(n^2 + n + 1) \geqq 2$. Result **12** was the starting point for a more thorough investigation of Bose & Shrikhande 1959, 1960; Parker 1959b; and Bose, Shrikhande & Parker 1960. In the last paper it is shown that

$$N(n) > 1 \qquad \textit{for all } n \neq 1, 2, 6. \tag{21}$$

---

[1] Euler 1782 investigated the "problem of the 36 officers": *Is it possible that 36 officers, of 6 different ranks and from 6 different regiments, stand in a square of 6 rows and 6 columns in such a way that every rank and every regiment is represented exactly once in every row and every column?* This is clearly equivalent to finding two orthogonal Latin squares of order 6. Euler conjectured correctly that the answer to this problem is negative (Tarry 1900), and incorrectly [see (21) below] that the corresponding problem for $n \equiv 2 \bmod 4$, instead of $n = 6$, is also unsolvable [Euler 1782, p. 183, § 144]. His conjecture was generalized by MacNeish 1922; see the context of (20) below. Euler was the first to prove that $N(n) > 1$ if $n \not\equiv 2 \bmod 4$.

Note that $N(1) = N(2) = 1$ is trivial; $N(6) = 1$ was proved by TARRY 1900, 1901. In the papers of Bose, Shrikhande and Parker just mentioned, much more than (21) is shown, but for these and other results on Latin squares the reader must be referred to the literature.[1]

The last results gave lower bounds for the function $N(n)$. There are also some upper bounds, distinct from the trivial **11a**: Result **2.1.15,** when restricted to the case $\lambda = 1$, gives the following nonexistence theorem for finite planes:

13. *If* $n \equiv 1$ *or* 2 *mod* 4 *and if* n *is not the sum of two squares* [*i.e. if the square-free factor of* n *has a prime divisor* $p \equiv 3$ mod 4],[2] *then there exists no* (*projective or affine*) *plane of order* n.

This is the celebrated Theorem of BRUCK & RYSER 1949. It follows immediately that no finite plane has an order $\equiv 6$ mod 8. In view of **9, 10** and **13,** we can now conclude:

14. *If* n *satisfies the conditions of* **13,** *then*

$$n \leqq f(n-2-N(n)) < \tfrac{1}{2}(n-1-N(n))^4, \tag{22}$$

where f *is the polynomial defined by* (17).

(BRUCK 1963a.) Results **10, 11b** and **13** constitute our complete knowledge on the existence problem of finite projective planes with prescribed order. All known planes have prime power order. The smallest orders for which the existence problem is undecided are $n = 10, 12, 15, 18, 20, 24, 26, 28, 34, 35, 36$.

We shall see in Chapter 5 that for any $n = p^e > 8$, with p a prime and $e > 1$, there exist at least two nonisomorphic projective planes of order n. Whether this is true for $e = 1$ also is an open problem. But for small orders there is at most one projective plane:

15. *Every projective plane of order* $n \leqq 8$ *is desarguesian, i.e. isomorphic to a* $\mathbf{P}(q)$ *as defined in Section* 1.4.

For $n \leqq 5$, this was proved by MACINNES 1907; see also PICKERT 1955, p. 302. By **13,** there is no plane of order 6. For the uniqueness of $\mathbf{P}(7)$, cf. BOSE & NAIR 1941; HALL 1953, 1954b; PIERCE 1953; PICKERT 1955, pp. 319—325. The uniqueness of $\mathbf{P}(8)$ was determined by an

---

[1] We mention here the following references: LEVI 1942; MANN 1942, 1943, 1950; CHOWLA, ERDÖS & STRAUS 1960. Latin squares can be interpreted as multiplication tables of finite quasigroups; for results in this context see JOHNSON, DULMAGE & MENDELSOHN 1961, also BRUCK 1958. Complete tables of Latin squares are given by TARRY 1900, 1901 and PETERSEN 1902 for $n = 6$, and by NORTON 1939 (with omissions) and SADE 1951 for $n = 7$.

[2] The equivalence of these conditions is well known; see for example HARDY & WRIGHT 1962, p. 299.

electronic computer; see Hall, Swift, Walker 1956. That there exist nondesarguesian planes of order 9 (three such planes are known) will be seen in Chapter 5.

Despite the existence of non-desarguesian planes, it can be shown that every finite projective plane contains many Desargues configurations, i.e. subsystems of points and lines isomorphic to the incidence structure shown in Fig. 1 (p. 26). More precisely:

16. *Let* (c, A) *be a point-line pair in a finite projective plane* $\mathbf{P}$, *let* M_i $(i = 1, 2, 3)$ *be three lines* $\neq A$ *through* c, *and let* r_1, r_2 *be two points* $\neq AM_i$ $(i = 1, 2, 3)$ *on* A. *Then* $\mathbf{P}$ *contains a Desargues configuration in which* c, A, M_i, r_j *have the same significance as in Fig.* 1.

This was proved by a simple but ingenious counting argument, by Ostrom 1957.

Few general results are known about the number of points necessary to generate a finite projective plane [cf. (3.1.10)]; this is known only in the desarguesian case:

17. *The desarguesian projective plane* $\mathbf{P}(q)$ *is generated by any one of its quadrangles (i.e. it is a prime plane) if, and only if,* q *is a prime. If* q *is not a prime, then* $\mathbf{P}(q)$ *can be generated by any quadrangle* $\mathfrak{q}$ *and a suitable point on one of the sides of* $\mathfrak{q}$.

This is a simple consequence of the fact that the multiplicative group of a finite field is cyclic.

It is conceivable that every nondesarguesian finite projective plane (i) can be generated by some quadrangle and (ii) contains the seven-point plane $\mathbf{P}(2)$ as a subplane. This has been found true in all planes which have been investigated in this respect; see Wagner 1956, Cofman 1964, Killgrove 1964, for (i), and Lenz 1953, H. Neumann 1955, for (ii). On the other hand, the existence of too many subplanes $\mathbf{P}(2)$ implies Desargues' theorem (Gleason 1956; cf. result **3.4.23** below).

We turn to a brief discussion of the possible orders of subplanes of a finite projective plane. The following is a useful lemma:

18. *Let* $\mathbf{P}$ *be a projective plane of order* n, *and* $\mathbf{Q}$ *a proper subplane of order* m. *Then*

(a) $m^2 = n$ *if and only if* $\mathbf{Q}$ *is a Baer subplane,*
and
(b) $m^2 + m \leqq n$ *if* $\mathbf{Q}$ *is not a Baer subplane.*

The proof is again by simple counting; see Bruck 1955, Lemma 3.1, and Hall 1959, p. 398. It is an unsolved problem whether **(b)** can hold with equality; clearly this would imply the existence of planes of orders

$m(m+1)$ which are not prime powers.[1] We shall see in Section 4.1 that $m^2 + m \neq n$ at least in the case where **Q** is the system of fixed elements of some collineation group.

A little more information on subplanes of finite planes can be obtained by considering the tactical decomposition defined by a subplane (Dembowski 1958, Section 2.4). We shall not present this here, but for the convenience of the reader, and for later reference, we restate the basic equations for tactical decompositions, viz. (1.1.14—16) and (2.1.16) and its dual, for the case the incidence structure under consideration is a projective plane of order n. If $\mathfrak{x}$ is a point class and $\mathfrak{X}$ a line class, then $(\mathfrak{x}\,\mathfrak{X})$ denotes the number of points of $\mathfrak{x}$ on any line of $\mathfrak{X}$, and $(\mathfrak{X}\,\mathfrak{x})$ has the dual meaning. Then:

$$(\mathfrak{x}\,\mathfrak{X})\,|\mathfrak{X}| = (\mathfrak{X}\,\mathfrak{x})\,|\mathfrak{x}| \qquad \textit{for all } \mathfrak{x},\ \mathfrak{X}; \tag{23}$$

$$\sum_{\mathfrak{x}} |\mathfrak{x}| = \sum_{\mathfrak{X}} |\mathfrak{X}| = n^2 + n + 1; \tag{24}$$

$$\sum_{\mathfrak{x}} (\mathfrak{x}\,\mathfrak{C}) = \sum_{\mathfrak{X}} (\mathfrak{X}\,\mathfrak{c}) = n + 1 \qquad \textit{for all } \mathfrak{c}, \mathfrak{C}; \tag{25}$$

$$\sum_{\mathfrak{X}} (\mathfrak{c}\,\mathfrak{X})\,(\mathfrak{X}\,\mathfrak{c}') = |\mathfrak{c}| + n\,\delta(\mathfrak{c}, \mathfrak{c}') \qquad \textit{for all } \mathfrak{c}, \mathfrak{c}'; \tag{26}$$

$$\sum_{\mathfrak{x}} (\mathfrak{C}\,\mathfrak{x})\,(\mathfrak{x}\,\mathfrak{C}') = |\mathfrak{C}| + n\,\delta(\mathfrak{C}, \mathfrak{C}') \qquad \textit{for all } \mathfrak{C}, \mathfrak{C}'. \tag{26'}$$

Here $\delta(x, y) = 1$ or 0 according as $x = y$ or $x \neq y$.

We mention a rather isolated application, showing that the concept of homomorphism is of little value for finite planes.[2] In fact:

19. *Every epimorphism φ of a projective design* **D** *onto a projective plane* **P** *is an isomorphism.*[3]

Hence **P** and **D** are then isomorphic projective planes, and in particular there exist no proper epimorphisms of one finite projective plane onto another.[4] The proof of **19** appears in Dembowski 1959; it is shown there that the *cosets* of φ, i.e. the pre-images of the elements in **P** under φ, must form a tactical decomposition Δ of **D** such that the quotient structure **D**/Δ, as defined in the end of Section 1.1, is isomorphic to **P**. But:

---

[1] An interesting but so far unsuccessful idea for the construction of such planes, with the help of certain systems of 1-spreads of $\mathcal{P}(3, m)$ (for definitions see Section 1.4), has been pointed out by Bruck 1963b, Section 9.

[2] It is true that every projective plane is a homomorphic image of a *free plane* (cf. Hall 1943, or Pickert 1955, Section 1.3) but this fact has, so far, not yielded much for the theory of finite planes.

[3] The definitions for epi- and isomorphisms of incidence structures are in Section 1.2. Result **19** is in contrast to some theorems in Section 7.2 below: if a certain generalization of the concept of "design" is admitted, then there do exist proper homomorphisms onto finite projective planes.

[4] This more special result was also proved by Hughes 1960b and Corbas 1964.

20. *If a projective design* $\mathbf{D}$ *has a tactical decomposition* Δ *such that* $\mathbf{D}/\Delta$ *is a projective plane* $\mathbf{P}$, *then* Δ *is the trivial decomposition each of whose classes has only one element, and hence* $\mathbf{D} \cong \mathbf{P}$.

This is Satz 6 of Dembowski 1958, and proves **19**.

For the remainder of this section, we shall be concerned with point sets in projective planes which contain no three collinear points. Such a set will be called an *arc*, and an arc which consists of k points will be called a *k-arc*. For example, the 3- and 4-arcs are just the triangles and quadrangles. An arbitrary line L meets an arc $\mathfrak{c}$ in at most two points; L will be called *secant, tangent,* or *exterior* according as $|(L) \cap \mathfrak{c}| = 2, 1$, or 0. Note that

21. *In a projective plane of order* n, *every k-arc* $\mathfrak{c}$ *has* $\binom{k}{2}$ *secants,* $k(n+2-k)$ *tangents, and* $\binom{n}{2} + \binom{n+2-k}{2}$ *exterior lines. Through every point of* $\mathfrak{c}$ *there are* $k-1$ *secants and* $n+2-k$ *tangents.*

An *oval*[1] is an arc $\mathfrak{o}$ of a (not necessarily finite) projective plane such that

(27) *Through every point of* $\mathfrak{o}$, *there is exactly one tangent.*

Hence **21** implies:

22. *The ovals in finite planes of order* n *are precisely the* $(n+1)$*-arcs.*

Every arc in a finite projective plane gives rise to a system of diophantine equations which will now be displayed.[2] Let $\mathfrak{c}$ be a fixed k-arc in a projective plane of order n. For any point $x \notin \mathfrak{c}$, define $t(x)$ as the number of tangents of $\mathfrak{c}$ passing through x. The number of secants through x is then clearly $[k - t(x)]/2$, whence

$$(28) \qquad t(x) \equiv k \bmod 2, \qquad \textit{for every point } x \notin \mathfrak{c}.$$

Now define e_i as the number of those points $x \notin \mathfrak{c}$ for which $t(x) = i$; simple counting then yields the following equations:

$$(29) \qquad \left\{ \begin{aligned} \sum_{i=0}^{k} e_i &= n^2 + n + 1 - k, \\ \sum_{i=0}^{k} i\, e_i &= k\, n(n + 2 - k), \\ \sum_{i=0}^{k} i(i-1)\, e_i &= k(k-1)\,(n + 2 - k)^2. \end{aligned} \right.$$

---

[1] This is what we have called an "ovoid" in Section 1.4. The term "oval" is more customary for projective planes. The reader must be warned that our terminology differs somewhat from that of Segre 1961.

[2] These may be interpreted as consequences of (23)—(26′), for a certain tactical decomposition associated with a k-arc by Lombardo-Radice 1962. The special case where $k = n + 1$ or $n + 2$ is in Dembowski 1958, Section 2.5.

Furthermore, for any line L, let $f_i(L)$ denote the number of points $x \notin \mathfrak{c}$ on L for which $t(x) = i$. Then

$$(30) \qquad \sum_{i=0}^{k} f_i(L) = \begin{cases} n-1 & \textit{if } L \textit{ is secant,} \\ n & \textit{if } L \textit{ is tangent,} \\ n+1 & \textit{if } L \textit{ is exterior,} \end{cases}$$

and

$$(31) \qquad \begin{cases} \sum_{i=0}^{k} i\, f_i(L) = \begin{cases} (k-2)\,(n+2-k) & \textit{if } L \textit{ is secant,} \\ k(n+2-k) & \textit{if } L \textit{ is exterior,} \end{cases} \\ \sum_{i=0}^{k} (i-1)\, f_i(L) = (k-1)\,(n+2-k) \qquad \textit{if } L \textit{ is tangent.} \end{cases}$$

Furthermore,

$$(32) \quad \sum_{L} f_i(L) = \begin{cases} (k-i)e_i/2 \\ i\, e_i \\ [n+1-(k+i)/2]\, e_i \end{cases} \textit{if } L \textit{ ranges over all} \begin{cases} \textit{secants} \\ \textit{tangents} \\ \textit{exterior lines,} \end{cases}$$

for $i = 0, \ldots, k$. The proofs for (29)—(32) are all given by MARTIN 1967a; some of these equations are also in SEGRE 1959a, b; 1961, Chapter 17. See also BARLOTTI 1965.

An arc $\mathfrak{c}$ of the projective plane **P** is called *complete* if it is not properly contained in another arc of **P** or, equivalently, if every point of **P** is on a secant of $\mathfrak{c}$. In the finite case, we can say that a k-arc is complete if and only if $e_k = 0$. The following completeness results may all be proved from (28)—(32), the main tool being (29).

23. *Let* $\mathfrak{o}$ *be an oval in a projective plane of order* n. *Then there are* $n+1$ *tangents, one through each point of* $\mathfrak{o}$, $\binom{n+1}{2}$ *secants, and* $\binom{n}{2}$ *exterior lines. Moreover,*

(a) *If* n *is odd, then* $\mathfrak{o}$ *is complete, with* $e_2 = \binom{n+1}{2}$, $e_0 = \binom{n}{2}$, *and* $e_i = 0$ *for* $i \neq 0, 2$.
(b) *If* n *is even, then* $\mathfrak{o}$ *is not complete; in fact* $e_{n+1} = 1$, $e_1 = n^2 - 1$, *and* $e_i = 0$ *for* $i \neq 1, n+1$.

This result is due to QVIST 1952. It shows that the concept of an oval is self-dual if n is odd: the tangents then form an oval in the dual plane. This is not so, however, if n is even: in that case there exists a unique point $k = k(\mathfrak{o})$, the *knot* of the oval $\mathfrak{o}$, which is on all the $n+1$ tangents of $\mathfrak{o}$, and conversely every line through $k(\mathfrak{o})$ is a tangent. The set $\mathfrak{o} \cup k(\mathfrak{o})$ is then a (necessarily complete) $(n+2)$-arc. As a consequence, we note:

24. *If a plane of order n contains a k-arc, then*

$$k \leqq \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n+2 & \text{if } n \text{ is even.} \end{cases}$$

This was first proved by BOSE 1947a.

The following results show that an oval $\mathfrak{o}$ and a k-arc $\mathfrak{c} \nsubseteqq \mathfrak{o}$ cannot have too many common points:

25. *Let $\mathfrak{o}$ be an oval and $\mathfrak{c}$ a k-arc in a projective plane of order n.*

(a) *If n is odd, suppose that $|\mathfrak{c} \cap \mathfrak{o}| > (n+3)/2$ or that $|\mathfrak{c} \cap \mathfrak{o}| > (n+1)/2$ and $k > 3(n+3)/4$. Then $\mathfrak{c} \subseteqq \mathfrak{o}$.*

(b) *If n is even, suppose that $|\mathfrak{c} \cap (\mathfrak{o} \cup \{k(\mathfrak{o})\})| > (n+2)/2$. Then $\mathfrak{c} \subseteqq \mathfrak{o} \cup \{k(\mathfrak{o})\}$.*

For the proof, see MARTIN 1967a; a slightly weaker version is in BARLOTTI 1965, Section 2.5, and the case where $\mathfrak{c}$ is also an oval is contained in QVIST 1952. LOMBARDO-RADICE 1956 has exhibited a class of complete $[(q+3)/2]$-arcs in $\mathbf{P}(q)$, where q is a prime power $\equiv 3 \bmod 4$; this shows that the first result of **25a** is best possible. The second result of **25a** is also best possible, but there cannot exist examples showing this in a desarguesian finite projective plane: SEGRE's [1954, 1955a] Theorem **1.4.50** states that every oval in a desarguesian finite projective plane of odd order is a conic, and it is well known that, in any projective geometry over a commutative field, a conic is determined by five points. However, there exist two ovals in a nondesarguesian plane[1] of order 9 with five common points (MARTIN 1967a). Other examples of ovals in nondesarguesian planes appear in WAGNER 1959; RODRIGUES 1959; BARLOTTI 1965, Section 2.9.

Several papers, mostly by Italian authors, have been devoted to completeness criteria for k-arcs with $k < n+1$; the principal reference for these is Chapter 17 of SEGRE 1961. We survey some of these results here. It is not difficult to see that

26. *k-arcs with $\binom{k-1}{2} < n$ are not complete;*

and for small k this may be further improved: 5-arcs are never complete, 4-arcs are incomplete if $n > 3$, 6-arcs are incomplete if $n > 10$, and 7-arcs are incomplete if $n > 13$. As completeness is equivalent to $e_k = 0$, it is desirable to have inequalities involving e_k. In this context

---

[1] This plane, the "smallest Hughes plane", will be defined and discussed in Section 5.4.

we mention SCE's [1958] inequalities:

$$(33)\quad \frac{(k-1)(k-2)^2}{2} \leqq e_k - n^2 + \frac{n(k-1)(k-2)}{2}$$
$$\leqq \frac{(k-1)(k-2)}{2}\left[\frac{k(k-3)}{4}+1\right]$$

for any k-arc in a projective plane of order n.

An arc is called *uniform* if there is at most one $i > 2$ for which $e_i \neq 0$. Ovals and $(n+2)$-arcs in planes of order n are uniform.

27. *Uniform k-arcs in planes of order n are complete, except when $k = 4$ or n if k is even, or when $k = 5$ or $n - 1$ if k is odd.*

For the proof, see MARTIN 1967a; there exist counterexamples in the excepted cases.

Finally, we mention briefly n-arcs in planes of order n. SEGRE 1955b has shown that

28. *In the desarguesian plane* $\mathbf{P}(q)$ *of order q, there exists no complete q-arc.*

For the proof, see also SEGRE 1961, no. 175. The result cannot be extended to nondesarguesian planes: there exist complete 9-arcs in a nondesarguesian projective plane[1] of order 9 which are uniform with $e_5 = 6$, and hence complete by **27**; see BARLOTTI 1965, Section 2.9, and MARTIN 1967a who also gives various conditions necessary and sufficient for an n-arc to be complete.

The concept of k-arc has been generalized to that of (k, m)*-arc*; these are point sets $\mathfrak{c}$ with $|\mathfrak{c}| = k$ containing m, but not $m+1$, collinear points [hence k-arcs are $(k, 2)$-arcs]. We do not discuss these here, but mention only three references: BARLOTTI 1956; COSSU 1961; and BARLOTTI 1965, Chapter 3, where there are also more references.

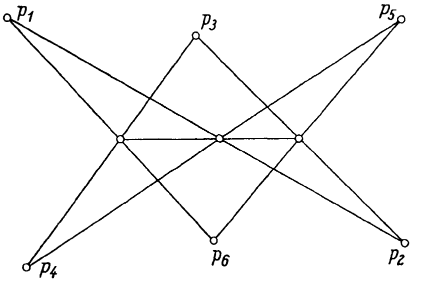

Fig. 3 Pascalian hexagon

In conclusion, we mention a beautiful theorem of BUEKENHOUT 1966b on pascalian ovals which holds also in the infinite case. By a

---

[1] See footnote [1] on p. 149.

hexagon, in an arbitrary projective plane **P**, we mean a sextuple $(p_1, \ldots, p_6)$ of distinct points such that the lines $p_i p_{i+1}$ (subscripts mod 6) are all distinct. A hexagon is *pascalian* if the three points $(p_i p_{i+1})$ $(p_{i+3} p_{i+4})$, $i = 1, 2, 3$ reduced mod 6, are collinear (see Fig. 3). The classical Theorem of Pascal says that in a projective plane over a commutative field, a hexagon is pascalian if it is contained in a conic, i.e. a nondegenerate quadric in the language of Section 1.4. Buekenhout's theorem is the following extension of this result:

29. *Suppose that the projective plane* **P** *contains an oval* $\mathfrak{o}$ *such that every hexagon of points in* $\mathfrak{o}$ *is pascalian. Then* **P** *is the desarguesian plane over some commutative field* $\mathfrak{K}$, *and* $\mathfrak{o}$ *is a conic in* **P**.

The main ideas of the proof are the following. First it is shown that the permutations $\sigma(p)$ of $\mathfrak{o}$, defined for every point $p \notin \mathfrak{o}$ by

$$(34)\quad x^{\sigma(p)} = \begin{cases} \textit{the point} \neq x \textit{ of } \mathfrak{o} \textit{ on } px, & \textit{if } px \textit{ is secant to } \mathfrak{o}, \\ x & \textit{if } px \textit{ is tangent to } \mathfrak{o}, \end{cases}$$

generate a sharply 3-transitive group Σ of permutations of $\mathfrak{o}$. The condition that the hexagons in $\mathfrak{o}$ are pascalian also implies that Σ is faithfully induced by a collineation group of **P** which may also be called Σ. This allows the reconstruction of **P** within Σ; for example, the points $\notin \mathfrak{o}$ of **P** may be identified with the involutions. On the other hand, a result of Tits 1952 shows that $\Sigma \cong PGL_2(\mathfrak{K})$ for some commutative field $\mathfrak{K}$. If **Q** is the desarguesian plane over $\mathfrak{K}$ and $\mathfrak{c}$ any conic in **Q**, one may consider the permutations (34) with $\mathfrak{c}$ and **Q** instead of $\mathfrak{o}$ and **P**; the resulting group is well known to be $PGL_2(\mathfrak{K})$ also. Thus the isomorphy of Σ and $PGL_2(\mathfrak{K})$ permits us to set up an isomorphism from **P** onto **Q** which maps $\mathfrak{o}$ onto $\mathfrak{c}$. For more details, the reader is referred to Buekenhout 1966b.

3.3 Correlations and polarities

By a *correlation* is meant here an anti-automorphism of a projective plane $\mathbf{P} = (\mathfrak{p}, \mathfrak{L}, \mathrm{I})$, i.e. a permutation ϱ of $\mathfrak{p} \cup \mathfrak{L}$ such that $\mathfrak{p}\varrho = \mathfrak{L}$, $\mathfrak{L}\varrho = \mathfrak{p}$, and $p \mathrel{\mathrm{I}} L$ if and only if $p\varrho \mathrel{\mathrm{I}} L\varrho$ for all $p \in \mathfrak{p}$, $L \in \mathfrak{L}$ (cf. Section 1.2). A *polarity* is a correlation of order 2. Clearly, a correlation ϱ which is not a polarity has the property that ϱ^2 is a nontrivial collineation.

In this section, we consider correlations, and mostly polarities, of finite projective planes. Let **P** be such a finite plane, and write its order as

$$(1)\qquad n = n^* s^2,$$

where n^* is square-free. We denote by $\mathfrak{a} = \mathfrak{a}(\varrho)$ and $\mathfrak{A} = \mathfrak{A}(\varrho)$ the sets of absolute points and lines, respectively, of the correlation ϱ, and by $\mathbf{A} = \mathbf{A}(\varrho)$ the substructure of $\mathbf{P}$ defined by $\mathfrak{a}$ and $\mathfrak{A}$; see (1.1.4) for definitions. Obviously $|\mathfrak{a}| = |\mathfrak{A}|$; we put

$$a(\varrho) = |\mathfrak{a}(\varrho)| = |\mathfrak{A}(\varrho)|. \tag{2}$$

The starting point of our discussion is the following corollary (for $\lambda = 1$) of result **2.1.17**:

1. *Every correlation ϱ of a finite projective plane has absolute points. In fact,*

$$a(\varrho) \equiv 1 \mod n^* s. \tag{3}$$

Moreover, if ϱ is a polarity, then there exists a nonnegative integer r such that

$$a(\varrho) = n + 1 + 2r\sqrt{n}. \tag{4}$$

Equations (3) and (4) are due to BALL 1948 and HOFFMAN, NEWMAN, STRAUS, TAUSSKY 1956. That $r \geqq 0$ is not immediate from **2.1.17**, but it is not difficult to prove (for example, with the help of **5** below). We note the following consequences of (4):

2. *Let π be a polarity of a projective plane of order n. Then*

$$a(\pi) \geqq n + 1; \tag{5}$$

equality holds whenever n is not a square.

This was first proved by BAER 1946a, Section 1, Theorems 5 and 6.

Note that **2** implies the nonexistence of collineation groups of Lenz-Barlotti type I.8 in finite projective planes; cf. **3.1.20.**1)

A power ϱ^i of a correlation ϱ is, of course, a collineation or a correlation according as i is even or odd. The case where i is even will be discussed at the end of Section 4.1; for i odd we have the following result:

3. *Let ϱ be a correlation of a plane of order n. For any odd prime p not dividing n, for any odd integer j prime to p, and for any integer $i \geqq 0$,*

$$\textit{if} \begin{Bmatrix} \left(\frac{n}{p}\right) = 1 \\ \left(\frac{n}{p}\right) = -1 \end{Bmatrix}, \ \textit{then} \ a(\varrho^{jp^i}) \equiv \begin{Bmatrix} a(\varrho^{jp^{i+1}}) \\ 2(n+1) - a(\varrho^{jp^{i+1}}) \end{Bmatrix} \mod p^{i+1}.$$

Here $\left(\frac{n}{p}\right)$ is the Legendre symbol. For the proof of **3**, see BALL 1948, Proposition 2.1. From this, and from the Dirichlet theorem that there are infinitely many primes of the form $ex + d$ whenever e and d are relatively prime [see, for example, LEVEQUE 1956, vol. II, Chapter 6], Ball has derived the following result:

---

1) On the other hand, it will be shown in Section 4.3 that in the infinite case such groups cannot exist either.

4. *If n is a square and i relatively prime to the order of ϱ (which implies that i is odd), then $a(\varrho) = a(\varrho^i)$.*

(BALL 1948, Theorem 3.1) Ball also gives some results for the case where n is not a square (Theorem 3.2 of the quoted paper). Some of Ball's results have been proved in another way by HOFFMAN, NEWMAN, STRAUS, TAUSSKY 1956; these authors have also given more conditions for $a(\varrho) = n + 1$. We remark further that (5) is not true for arbitrary correlations; an example with $a(\varrho) = 1$ appears in BALL 1948, p. 931.

For the remainder of this section, let π be a polarity of the projective plane **P** of order n. Result **1.2.2** shows that every absolute point (line) of π is incident with exactly n non-absolute lines (points) of π. If L is a non-absolute line, then the mapping $x \rightarrow (x\pi)L$ is a well-defined involutorial permutation of the point set (L), and the fixed points of this permutation are just the absolute points of π on L. Hence:

5. *The number of non-absolute points (lines) incident with a non-absolute line (point) is even.*

From this and $a(\pi) > 0$ it is not difficult to derive (5); cf. the remark after **1** above. Furthermore, if (5) holds with equality, **5** yields satisfactory information about the substructure $\mathbf{A}(\pi)$ of the π-absolute elements of **P**:

6. *Let π be a polarity with $a(\pi) = n + 1$, of a projective plane of order n.*

(a) *If n is odd, then $\mathbf{A}(\pi)$ consists of the points and tangents of some oval.*

(b) *If n is even, then $\mathbf{A}(\pi)$ consists of the points of a distinguished non-absolute line L and the lines through $L\pi$.*

(BAER 1946a, p. 82, Corollary 1.)

Next, we restate some results of Section 1.4, on polarities of finite desarguesian projective planes:

7. *Let π be a polarity of the desarguesian projective plane $\mathbf{P}(q)$ of order $q = p^e$. Then either $a(\pi) = q + 1$ (if π is orthogonal) or $a(\pi) = q^{3/2} + 1$ (if π is unitary). In the second case, which cannot occur unless $q = s^2$ is a square, the substructure of the absolute points and non-absolute lines of π is a design with parameters*

$$(6) \qquad v = s^3 + 1, \quad b = s^2(s^2 - s + 1), \quad k = s + 1,$$
$$r = s^2 = q, \quad \lambda = 1.$$

This substructure is clearly what we have called a unital[1] in Section 2.4.

---

[1] Compare here (2.4.20) and context, where this unital was denoted by $\mathbf{U}(s)$, and the remarks preceding **1.4.60**. For a generalization, see **9** below.

We discuss now a more general situation where $\mathbf{P} = (\mathfrak{p}, \mathfrak{L}, \mathrm{I})$ is finite of order n but not necessarily desarguesian. We call the polarity π of $\mathbf{P}$ *regular* (with BAER 1946a) if, for some integer $s = s(\pi)$, the number of absolute points on a non-absolute line (absolute lines through a non-absolute point) is either 0 or $s + 1$. We can then restate **7** as follows: every polarity of a desarguesian finite projective plane is regular, and s is either 1 or $\sqrt{n}$. For an arbitrary regular polarity, the non-absolute lines fall into two disjoint classes: the set $\mathfrak{O} = \mathfrak{O}(\pi)$ of the *outer lines* which carry no absolute point, and the set $\mathfrak{J} = \mathfrak{J}(\pi)$ of the *inner lines*, each carrying $s + 1$ absolute points. Dually, $\mathfrak{i} = \mathfrak{i}(\pi)$ is the set of *inner points*, carrying no absolute line, and $\mathfrak{o} = \mathfrak{o}(\pi)$ is the set of *outer points*, each carrying $s + 1$ absolute lines. Note that $\mathfrak{O}\pi = \mathfrak{i}$ and $\mathfrak{J}\pi = \mathfrak{o}$. It is almost immediate that the classes $\mathfrak{O}$, $\mathfrak{J}$, $\mathfrak{o}$, $\mathfrak{i}$, together with the classes $\mathfrak{A} = \mathfrak{A}(\pi)$ and $\mathfrak{a} = \mathfrak{a}(\pi)$ of the absolute elements, form a tactical decomposition of $\mathbf{P}$, in the sense defined in Section 1.1 Using the basic equations (3.2.23)—(3.2.26′) for tactical decompositions of finite projective planes, and putting

$$\mathfrak{p}_1 = \mathfrak{a}, \quad \mathfrak{p}_2 = \mathfrak{o}, \quad \mathfrak{p}_3 = \mathfrak{i}, \quad \mathfrak{B}_1 = \mathfrak{A}, \quad \mathfrak{B}_2 = \mathfrak{J}, \quad \mathfrak{B}_3 = \mathfrak{O},$$

one derives the matrix $C = ((\mathfrak{p}_i\, \mathfrak{B}_j))$, defined by (1.3.1):

$$C = \begin{pmatrix} 1 & s+1 & 0 \\ n & \dfrac{s(n-1)}{s+1} & \dfrac{sn+1}{s+1} \\ 0 & \dfrac{n-s^2}{s+1} & \dfrac{n+s}{s+1} \end{pmatrix}; \tag{7}$$

furthermore:

$$(8)\quad a(\pi) = sn + 1, \quad |\mathfrak{o}| = |\mathfrak{J}| = \frac{n(sn+1)}{s+1}, \quad \textit{and} \quad |\mathfrak{i}| = |\mathfrak{O}| = \frac{n(n-s^2)}{s+1}.$$

As all these numbers must be integers, we can conclude the following:

8. *Let π be a regular polarity of a projective plane of order n, and put $s(\pi) = s$. Then* (8) *holds, and*

$$s \equiv n \mod 2, \tag{9}$$

$$n - 1 \equiv 0 \mod s + 1, \tag{10}$$

$$1 \leqq s^2 \leqq n. \tag{11}$$

Also, $s = 1$ if n is not a square, $s^2 = n$ if n is even, and if $s^2 < n$, then $s^2 + s + 1 \leqq n$.

These results, which bear a certain resemblance to **3.2.18**, are due to BAER 1946a, Section 2. In analogy with the desarguesian case, we may call a polarity *unitary* if it is regular with $s^2 = n$. We can then infer from (8):

9. *The absolute points and nonabsolute lines of a unitary polarity of a finite projective plane form a unital.*[1])

It seems to be unknown whether there exist, necessarily in nondesarguesian finite planes, regular polarities with $1 < s^2 < n$. There do exist non-regular polarities; examples will be given in Section 5.3.

Two lines are called *perpendicular,*[2]) with respect to the polarity π, if each of them is incident with the pole of the other. Perpendicularity is clearly a symmetric relation, and any line is perpendicular to itself if and only if it is absolute. Also:

10. *If L is any line and $p \not\mathrel{I} L\pi$, then $p(L\pi)$ is the only line through p which is perpendicular to L.*

In the regular case, we call a point *elliptic*[2]) if it is the intersection of two inner lines which are perpendicular to each other. Consider the following conditions:

(12) *Every inner point is elliptic.*

(13) *No outer point is elliptic.*

(14) *If $p \mathrel{I} L$, with p elliptic and L inner, then the line through p perpendicular to L is also inner.*

The following result gives the relationships between these conditions:

11. *For any regular polarity π of a projective plane of order n,*

(a) $s(\pi) > 1$ *implies* (12).
(b) (13) *implies* (12), (14), $s(\pi) = 1$, *and* $n \equiv 3 \bmod 4$.
(c) (12), (14), *and* $s(\pi) = 1$ *together imply* (13).

This is essentially Theorem 6 of Baer 1946a, p. 88. Note that for unitary polarities (12) holds, but not (13).

We conclude this discussion with another result of Baer, showing that a projective plane with a certain familiar type of polarity is necessarily infinite. For every polarity π with the property that some line carries no absolute point, it is possible to divide the nonabsolute points into two disjoint sets $\mathfrak{i}$ and $\mathfrak{e}$ ("interior" and "exterior" points) such that

(15) $$\text{if } p \in \mathfrak{i} \text{ and } L \mathrel{I} p, \text{ then } L\pi \in \mathfrak{e}.$$

For example, $\mathfrak{i}$ may consist of a single point p carrying no absolute line, and $\mathfrak{e}$ of all other nonabsolute points. Now π is called *hyperbolic* if

---

[1]) Cf. footnote [1]) on p. 153.

[2]) We follow here (and further below) the terminology of Baer 1946a. See also Liebmann 1934.

such a division can be made in such a way that the following condition is also satisfied:

(16) *If* L *and* M *are perpendicular lines in* $\mathfrak{e}\pi$, *then* $LM \in \mathfrak{i}$.

The result then reads:

12. *No finite projective plane possesses a hyperbolic polarity.*

We outline a proof. (See Baer 1948 and, for a weaker preliminary version of the theorem, Baer 1946a, p. 91.) Assume that **P** is a projective plane with a hyperbolic polarity. Firstly, (15) and (16) show that both $\mathfrak{i}$ and $\mathfrak{e}$ are non-empty. Secondly, it follows easily from (15) and **1.2.2** that not all points on a line $L \in \mathfrak{e}\pi$ can be absolute. Hence every such L carries a non-absolute point p, and either p or $L(p\,\pi)$ is in $\mathfrak{i}$, by (16). Thus every line of $\mathfrak{e}\pi$ carries points of $\mathfrak{i}$. Next:

(17) *All lines in* $\mathfrak{e}\pi$ *carry equally many points of* $\mathfrak{i}$.

For let $L, M \in \mathfrak{e}\pi$. If L and M are non-perpendicular, then the mapping $x \to x' = L[x(M\pi)]$ is one-one and sends the points $x \in \mathfrak{i}$ on M onto the points $x' \in \mathfrak{i}$ on L [both (15) and (16) are used here]. If L and M are perpendicular, then $LM \in \mathfrak{i}$ by (16), and any line $X \mathrel{\mathrm{I}} LM$ is likewise in $\mathfrak{e}\pi$, by (15). If $\mathfrak{X} \neq L, M$, then X is perpendicular to neither L nor M, and both L and M carry the same number of points of $\mathfrak{i}$ as does X. This proves (17).

(18) *There are equally many points of* $\mathfrak{e}$ *and* $\mathfrak{i}$ *on any line of* $\mathfrak{e}\pi$.

To see this, define $x^0 = (x\,\pi)\,L$, for any $x \mathrel{\mathrm{I}} L \in \mathfrak{e}\pi$. Then $x^{00} = x$, and $x^0 = x$ if and only if x is absolute. Furthermore, (16) shows that if $x \neq x^0$, exactly one of x, x^0 is in $\mathfrak{e}$, the other in $\mathfrak{i}$. This proves (18).

So far, we have not used finiteness, so that (17) and (18) are true for any hyperbolic polarity. Now assume that **P** is finite of order n. From (17) and (18) we conclude that, for some integer m, every line of $\mathfrak{e}\pi$ carries exactly m points of $\mathfrak{e}$ and exactly m points of $\mathfrak{i}$. Then (15) shows that

$$|\mathfrak{i}| = 1 + (n+1)\,(m-1).$$

On the other hand, the dual of (15) shows that the n nonabsolute lines through an absolute point (see **5**) are in $\mathfrak{e}\,\pi$, and all points of $\mathfrak{i}$ are on these lines. Hence

$$|\mathfrak{i}| = n\,m,$$

and we conclude that $m = n$. But $2m \leqq n + 1$ by (18); this gives $n \leqq 1$, a contradiction proving **12.**

The polarity associated with the classical hyperbolic plane (Klein's model) is hyperbolic in our sense. Result **12** shows, therefore, that there is no finite analogue of this classical situation. In particular, a finite "hyperbolic plane" in the sense of GRAVES 1962 [cf. the remarks on p. 105], if it can be interpreted as the system of "interior points" and polars of "exterior points" with respect to a polarity of a finite projective plane, cannot satisfy condition (16).

The preceding results provide no information as to when a given finite projective plane actually admits a polarity. The following is a result in this direction which, incidentally, provides another proof that the desarguesian finite projective planes do possess polarities.

13. *A finite projective plane* **P** *which is* (v, W)*-transitive for some flag* (v, W) *admits a polarity* π *with* $W = v\pi$ *if, and only if, it can be coordinatized by a cartesian group* $\mathfrak{C}$ *which possesses an involutorial permutation* α *such that*

$$(19)\qquad (x+y)^\alpha = y^\alpha + x^\alpha \quad \textit{and} \quad (xy)^\alpha = y^\alpha x^\alpha.$$

In fact, any cartesian group $\mathfrak{C} = \mathfrak{T}(o, e, u, v)$ with

$$(20)\quad u \mathrm{I} W, \quad v\pi = W, \quad o\pi = ou, \quad \text{and} \quad (ev)\pi = (ov)W$$

will have this property, and if $\mathfrak{C}$ is a cartesian group satisfying (19), then the mapping π defined by

$$(21)\qquad (u, v)\pi = (y = xu^\alpha - v^\alpha)$$

is a polarity satisfying (20). For the proof of these results, see DEMBOWSKI & OSTROM 1968, Lemma 1. The following result is also proved in this paper (DEMBOWSKI & OSTROM 1968, Lemma 2):

14. *Let* $\mathfrak{C}$ *be a cartesian group of finite order* n *with commutative addition, and suppose that the plane over* $\mathfrak{C}$ *admits a polarity* π *satisfying* (20), *such that* $\mathfrak{A}(\pi)$ *is an oval. Then* n *is odd, and multiplication in* $\mathfrak{C}$ *is also commutative.*

In fact, it can be shown that the permutation α of (17) is the identity in this case.

3.4 Projectivities

Let (p, L) be a nonincident point-line pair in a projective plane **P**. We define a one-one mapping $\pi(L, p)$ of the set (L) of all points on L onto the set (p) of all lines through p by

$$(1)\qquad x^{\pi(L,p)} = xp \qquad \textit{for every } x \mathrm{I} L.$$

The inverse mapping will be denoted by $\pi(p, L)$:

$$(2)\qquad \pi(L, p)^{-1} = \pi(p, L).$$

A *projectivity*[1] of **P** is a mapping of some set (L) or (p) onto some other set (M) or (q) which can be written as a product of mappings (1)

---

[1] This term is often reserved only for mappings of lines onto lines, and later in this section we shall be only concerned with these.

and (2). Note that there are four distinct types of projectivities; here we shall consider mostly the type $(L) \to (M)$, which can be written in the form

$$\pi = \prod_{i=1}^{l} \pi(L_{i-1}, p_i)\, \pi(p_i, L_i), \tag{3}$$

where $L_0 = L$ and $L_l = M$. In order to avoid trivialities, we may assume also that

$$p_{i-1} \neq p_i \quad \textit{and} \quad L_{i-1} \neq L_i \quad (i = 1, \ldots, l-1), \tag{4}$$

because of (2). The integer l is called the *length* of the representation (3) of π, and if π cannot be represented by a product of type (3) with less than $2l$ factors, then l is also called the length of π. A projectivity of length 1 is called a *perspectivity.*

We state now some well known results on projectivities. For proofs, the reader is referred to VEBLEN & YOUNG 1910, Sections 23—26, and HESSENBERG 1930.

1. *If $p_i \mathrm{I} L$ and $q_i \mathrm{I} M$, where the p_i as well as the q_i are distinct $(i = 1, 2, 3)$, then there exists a projectivity π of length $\leqq 3$ such that $p_i^\pi = q_i$ $(i = 1, 2, 3)$. If $L \neq M$, such a projectivity exists with length $\leqq 2$.*

2. *If p_1, p_2, q_1, q_2 are four distinct points on L, then there exists a projectivity π of length $\leqq 3$ such that $p_1^\pi = p_2$, $p_2^\pi = p_1$ and $q_1^\pi = q_2$, $q_2^\pi = q_1$.*

In an arbitrary projective plane, the projectivities of **1** and **2** need not be unique. We shall now formulate a condition which guarantees this uniqueness.[1] A projective plane is called *pappian* if every hexagon $(p_1, \ldots, p_6)$ with $p_5 \mathrm{I} p_1 p_3$ and $p_6 \mathrm{I} p_2 p_4$ is pascalian,[2] in other words if the following condition is satisfied:

(5) *If $L \neq M$, if p_1, p_3, p_5 are distinct points on L and p_2, p_4, p_6 distinct points on M, and if $p_i \neq LM$ for $i = 1, \ldots, 6$, then*

$$q_1 = (p_1 p_2)(p_4 p_5), \quad q_2 = (p_2 p_3)(p_5 p_6) \quad \textit{and} \quad q_3 = (p_3 p_4)(p_6 p_1)$$

are collinear points (Fig. 4).

This condition is known as the "Theorem of Pappus"; it is the first in a sequence of *incidence propositions* to be considered in this section. The "Theorem of Desargues" [Condition (**D**) of Section 1.4] is another example.[3]

---

[1] The word "uniqueness" refers here to the actual mapping defined by a projectivity, not the manner of its representation: Many different products of the form (3) may define the same mapping.

[2] For the definition of pascalian hexagons, see the context of **3.2.29**.

[3] Instead of "incidence proposition", the terms "configuration theorem" and "Schliessungssatz" are customary. We avoid the use of the word "theorem" in this context.

The basic relationship between condition (5) and projectivities is expressed in the following lemma.

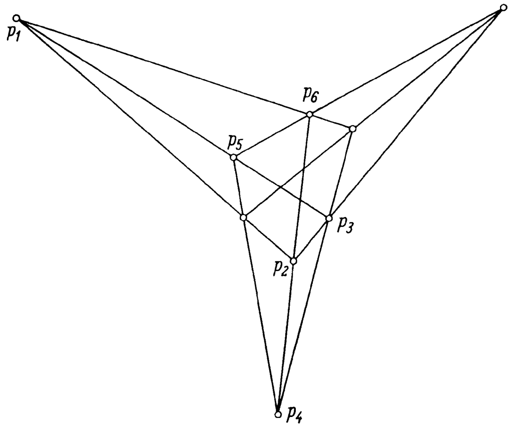

Fig. 4 Theorem of Pappus

3. *A projective plane is pappian if and only if it satisfies the following condition:*

(6) *Let* L_0, L_1, L_2 *be three nonconcurrent lines and suppose that the projectivity* $\pi = \pi(L_0, p_1)\,\pi(p_1, L_1)\,\pi(L_1, p_2)\,\pi(p_2, L_2)$ *satisfies* $(L_0 L_2)^\pi = L_0 L_2$. *Then* π *is equal to a perspectivity* $\pi(L_0, p_0)\,\pi(p_0, L_2)$.

The proof is straightforward; see, for example, HESSENBERG 1930, p. 68. Repeated application of **3** yields the following theorem:

4. *Every pappian projective plane is desarguesian.*

(HESSENBERG 1905) For the proof,[1] see CRONHEIM 1953 or PICKERT 1955, Sections 5.1, 5.2. Next, we translate (5) into conditions on central collineations:

5. *The following conditions for a projective plane* **P** *are equivalent:*

(**a**) **P** *is pappian.*
(**b**) **P** *is* (p, L)*-transitive for every point-line pair* p, L, *and for* $p \not{\mathrm{I}} L$ *the group of all* (p, L)*-homologies is abelian.*[2]
(**c**) **P** *is* (p, L)*-transitive for all* p, L, *and if* σ, τ *are homologies of* **P** *with the same axis but distinct centers, then* $\sigma^{-1}\tau^{-1}\sigma\tau$ *is an elation.*

The proof of this is straightforward. In view of **3.1.22i**, it follows from (**b**) that

---

[1] Hessenberg's original proof is not complete; he disregarded the possibility that certain additional incidences may occur in a Desargues configuration.

[2] That the group of (p, L)-elations $(p \mathrm{I} L)$ is abelian follows from **3.1.11**.

6. *A desarguesian projective plane is pappian if and only if its coordinatizing field is commutative.*

But every finite field is commutative (WEDDERBURN 1904, WITT 1931, ZASSENHAUS 1952, BRANDIS 1964), hence

7. *A finite projective plane is desarguesian if and only if it is pappian.*

We return to projectivities. If in (6) the word "nonconcurrent" is replaced by "concurrent", then (6) remains true even in every desarguesian plane. In view of **4**, we therefore have the following improvement of **3**:

8. *A projective plane is pappian if and only if, for any two distinct lines L and M, any projectivity of length $\leqq 2$ from (L) onto (M) which fixes LM is a perspectivity.*

With this result as the main tool, one can now prove the so-called "Fundamental Theorem of Projective Geometry", saying that in a pappian plane the projectivity π of **1** is unique, in other words that every projectivity in a pappian plane is determined by its action on three distinct points (or lines). We shall give an equivalent formulation here. The set of all projectivities of a line L onto itself is clearly a group, in any projective plane. If this group is called $\Pi(L)$ and if $M \neq L$, then

$$\Pi(M) = \pi^{-1}\,\Pi(L)\,\pi,$$

for any projectivity π from L on to M. Hence $\Pi(L)$ and $\Pi(M)$ are similar as permutation groups and in particular isomorphic. We can therefore write $\Pi(\mathbf{P})$ or simply Π instead of $\Pi(L)$; this group will be called the *group of projectivities* of $\mathbf{P}$. We can now summarize the results collected so far:

9. $\Pi(\mathbf{P})$ *is a triply transitive group, and it is sharply 3-transitive if and only if* $\mathbf{P}$ *is pappian.*

The second statement of **9** is equivalent to the "Fundamental Theorem" mentioned above. Little is known about Π if $\mathbf{P}$ is not desarguesian. In the finite case, triply transitive permutation groups are comparatively rare, and it seems plausible to conjecture that, if $\mathbf{P}$ is nondesarguesian of order n, then $\Pi(\mathbf{P})$ contains the alternating group of degree $n+1$. In a few special cases (with $n = 9, 16$) this was proved by BARLOTTI 1959, 1964. In particular, an example with $n = 16$ shows that Π need not be the full symmetric group of degree $n+1$.

We mention two recent improvements of **9**.

Let W be a fixed line of the projective plane $\mathbf{P}$, and let L be any line distinct from W. Denote by $\Pi^W(L)$ the group of all those permutations of $L - \{WL\}$ which are induced by projectivities (3) with $L_0 = L_l = L$ and $p_i \mathrm{I} W$ $(i = 1, \ldots, l)$. As in the case of $\Pi(L)$, this permutation group does not depend on the choice of the line $L \neq W$; it will therefore be denoted be $\Pi^W(\mathbf{P})$ or simply by Π^W.

10. *Let* $\mathbf{P}$ *be a projective plane which need not be finite.*

(a) *If* $\Pi_{uvxyz} = 1$ *for any five distinct points* $u, \ldots, z$, *then* $\Pi_{xyz} = 1$, *and* $\mathbf{P}$ *is pappian.*[1]

(b) *There exist three distinct points* x, y, z *such that* $\Pi^W_{xy} = \Pi^W_{xyz}$ *if, and only if,* $\mathbf{P}^W$ *is a translation plane with kernel* $\neq GF(2)$ [*cf.* **3.1.24**].

(c) *If there exist four distinct points* x, y, z, u *such that* $\Pi_{xyz} = \Pi_{xyzu}$, *then* $\mathbf{P}$ *is of Lenz-Barlotti type* VII.1 *or* VII.2, *and therefore pappian if finite.*

Result (**a**) is the main theorem of SCHLEIERMACHER 1967, and (**b**) is a relatively simple consequence of Theorem 1 of LÜNEBURG 1967b. As $\Pi^W(L)$ is in the stabilizer of WL in $\Pi(L)$, result (**c**) is a consequence of (**b**). In the finite case, we have:

11. *Let* $\mathbf{P}$ *be finite of order* n, *and suppose that* Π_x, *for an arbitrary point* x, *has a normal subgroup of order* n. *Then* $\mathbf{P}$ *is pappian.*

This was proved by LÜNEBURG 1967b, Theorem 3, for odd n and by YAQUB 1968 for even n. Yaqub's argument uses result **23** below.

We discuss now some connections between projectivities and collineations. Central collineations clearly induce perspectivities on nonfixed lines; hence:

12. *In a desarguesian projective plane, any projectivity of one line onto another is induced by a product of at most three central collineations.*

For a more detailed discussion, see PICKERT 1955, p. 114; also LENZ 1965, p. 30.

The following considerations, due to GLEASON 1956, give connections between projectivities and central collineations in a more general situation. Let L, X be two distinct lines, and $LX = c$. Also, let u, v be two points, not incident with L or X. Define

$$(7) \qquad \lambda_X(u, v) = \pi(L, u)\, \pi(u, X)\, \pi(X, v)\, \pi(v, L)$$

and put

$$(8) \qquad \Lambda_L(u, v) = \{\lambda_X(u, v) : c \,\mathrm{I}\, X \,\not{\mathrm{I}}\, u, v\}.$$

Then the following is easily verified:

13. $\Lambda_L(u, v)$ *is a set of permutations of* (L), *fixing* c *and* $(uv)L$, *and sharply transitive on the remaining points of* L. *Moreover,* $\Lambda_L(u, v)$ *is a group if and only if the following condition holds:*

---

[1] The analogue of this for six points is not true; cf. BARLOTTI 1964c.

(9) *Let* p_1, p_2 *be two distinct points* $\neq c, (uv)L$ *on* L, *and* X, Y *two distinct lines* $\neq L$ *and* $\not\mathrm{I}\ u, v$, *through* c. *Define* $x_i = X(p_i u), y_i = Y(p_i v)$, *and* $z_i = (x_i v)(y_i u)$, *for* $i = 1, 2$. *Then* c, z_1 *and* z_2 *are collinear* (cf. Fig. 5).

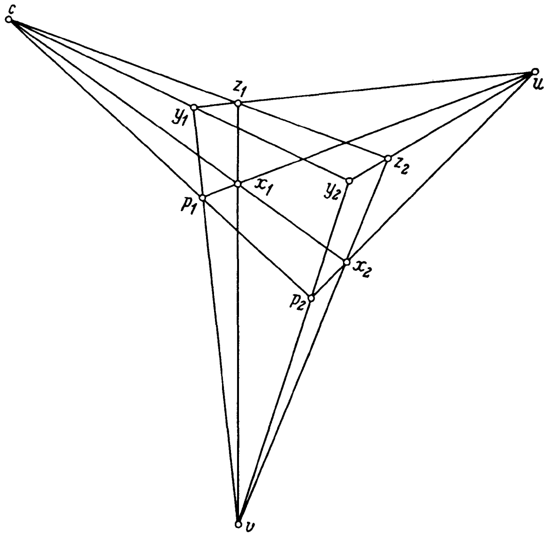

Fig. 5 Reidemeister condition

Condition (9) is called the *Reidemeister condition*, after REIDEMEISTER 1929. It is easily proved in any desarguesian plane[1], and conversely KLINGENBERG 1955 has shown that

14. *If the Reidemeister condition is satisfied in a projective plane* **P**, *then* **P** *is desarguesian.*

Combining results **13** and **14**, we can say that a projective plane is desarguesian if and only if each of the permutation sets $\Lambda_L(u, v)$ as defined in (8) is a group. The object of the following considerations is to give an essential improvement of this result in the finite case. We begin with some preparatory lemmas which are valid also in infinite planes.

15. *Let* (c, A) *be an arbitrary point-line pair in the projective plane* **P**, *and let* L *be an arbitrary line* $\neq A$ *through* c.

---

[1] As a matter of fact, the Reidemeister condition with $c \not\mathrm{I}\ uv$ is easily seen to be equivalent to associativity of multiplication, and that with $c\ \mathrm{I}\ uv$ to associativity of addition, in any ternary field of the plane under consideration.

(**a**) *A permutation of the point set* (L) *commutes with every permutation in the set*

$$\Lambda = \bigcup_{L \not{\mathrm{I}}\, u, v\, \mathrm{I}\, A} \Lambda_L(u, v)$$

if, and only if, it is induced by a collineation with center c *and axis* A.
(**b**) *If the sets* $\Lambda_L(u, v)$ *are groups for all* $u, v \neq c$ *on* A, *then*

$$(10) \qquad \Lambda_L(u, v) = \Lambda_L(v, u), \quad \textit{and}$$

$$(11) \qquad \Lambda_L(v, w) \subseteqq \Lambda_L(u, v)\, \Lambda_L(u, w) = \Lambda_L(u, w)\, \Lambda_L(u, v).$$

For the proof, see Gleason 1956, Lemma 2.1 and p. 805; the additional assumption $c \,\mathrm{I}\, A$ made there is unnecessary. These proofs are quite straightforward [for example, (**b**) follows from $\lambda_X(u, v) = \lambda_X(v, u)^{-1}$ and $\lambda_X(u, v)^{-1} \lambda_X(u, w) = \lambda_X(v, w)$]. We emphasize, however, that **15a** is of fundamental importance for the sequel.

16. **P** *is* (c, A)*-transitive if and only if* $\Lambda_L(u, v) = \Lambda_L(u, w)$, *for all lines* $L \neq A$ *through* c *and all points* $u, v, w \neq c$ *on* A.

This was proved by Kegel & Lüneburg 1963, p. 10, again under the superfluous condition $c \,\mathrm{I}\, A$. The equivalence of (c, A)-transitivity and (c, A)-Desargues is used here (cf. **3.1.16**).

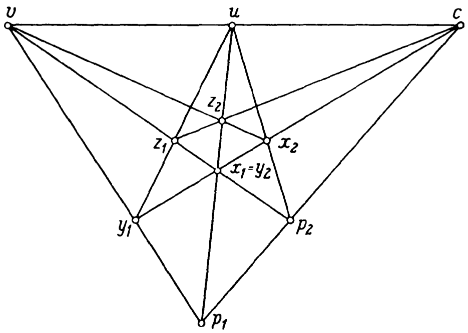

Fig. 6 Little hexagonality condition

We shall now be concerned with special cases of (9). The *little Reidemeister condition* is (9) with the additional assumption that u, v, c are collinear. By **13**, the little Reidemeister condition holds if and only if $\Lambda_L(u, v)$ is a group whenever $c \,\mathrm{I}\, uv$. The *hexagonality condition* is (9) with $x_1 = y_2$, and the *little hexagonality condition* (Fig. 6) is the hexagonality condition with $c \,\mathrm{I}\, uv$. It follows from theorems of Moufang 1931 that

17. *The little hexagonality condition is satisfied in a projective plane* **P** *if and only if every quadrangle in* **P** *generates a pappian prime subplane.*

For the proof, see PICKERT 1955, Chapter 11.[1]) Hence we can conclude that, in a plane satisfying the little Reidemeister condition, every quadrangle generates a subplane which is isomorphic either to a finite desarguesian plane $\mathbf{P}(p)$ for some prime p, or to the plane over the rational numbers.

Now we restrict ourselves to the finite case. Let $\mathbf{P}$ be a finite projective plane of order n, and suppose that the little Reidemeister condition is satisfied for $\mathbf{P}$. Then **17** implies that for every prime divisor p of n there exists a subplane isomorphic to $\mathbf{P}(p)$ in $\mathbf{P}$, and that every permutation in the group $\Gamma = \Lambda_L(u, v)$ [from now on always with $c \mathrm{I} uv$] has prime order. It follows that the cyclic subgroups of Γ form a *normal partition* of Γ, i.e. a set $\mathscr{C}$ consisting of full conjugate classes of subgroups, such that $\Gamma = \bigcup_{\Delta \in \mathscr{C}} \Delta$ and $\Delta \cap \Delta' = 1$ for any two distinct subgroups Δ, Δ' in $\mathscr{C}$ (cf. Section 1.2). For the following it is convenient to collect some results on normal partitions of finite groups:

18. *Let $\mathscr{C}$ be a normal partition of the finite group G.*

(a) *At most one conjugate class in $\mathscr{C}$ consists of subgroups which are their own normalizers in G.* [If there is such a conjugate class, $\mathscr{C}$ is called a *Frobenius partition.*[2])]

(b) *If $\mathscr{C}$ is a Frobenius partition and $F = F(G)$ the Fitting subgroup of G [i.e. the product of all nilpotent normal subgroups], then $X \in \mathscr{C}$ implies either $X \subseteqq F$ or $X \cap F = 1$ and $XF = G$. Also, $F \in \mathscr{C}$ unless F is a p-group.*

(c) *If G is not a p-group and $\mathscr{C}$ not a Frobenius partition, then every normal subgroup K satisfying $X \subseteqq K$ or $X \cap K = 1$ for all $X \in \mathscr{C}$ is itself in $\mathscr{C}$ and has index $[G:K]$ a prime divisor of $|K|$.*

(d) *If $X \in \mathscr{C}$ is not its own normalizer in G, then X is nilpotent.*

(e) *If G is non-soluble and $\mathscr{C}$ not a Frobenius partition, then G is isomorphic to $PGL_2(p^e)$ with odd p or $PSL_2(p^e)$ or $Sz(2^{2m+1})$, for suitable p, e, m.*

The proofs of **(a)**—**(c)** are in BAER 1961, Lemma 1.6, p. 343—345, and Satz 5.1. **(d)** is due to KEGEL 1961, Satz 2.[3]) Result **(e)** is the main theorem of SUZUKI 1961a; the symbol $Sz(q)$ stands, as on p. 52, for the Suzuki group over $GF(q)$, see SUZUKI 1960, 1962a, Section 13.

We return to the situation discussed above, and show:

---

[1]) In fact it is shown there that both conditions of **17** are equivalent to a special case of the theorem of Desargues (sometimes called condition D_8). In this context compare also DEMARIA 1959.

[2]) The reason for this is that the Frobenius groups (having a faithful transitive permutation representation which is not regular but in which only 1 fixes more than one symbol) are precisely the groups admitting such a partition.

[3]) This result will not be used here directly. We have listed it mainly because it is an important tool in the proof of **(e)**.

19. *If a finite projective plane satisfies the little Reidemeister condition, then its order is a power of a prime.*

Proof (Lüneburg 1961 b; Lüneburg & Kegel 1963): Let $\Gamma = \Lambda_L(u,v)$ for some line L and $uv \mathrm{I} c \mathrm{I} L$; cf. (8). Also, let $\mathscr{C}$ be the normal partition of Γ into its cyclic subgroups of prime order. Assume first that $\mathscr{C}$ is a Frobenius partition. If the Fitting subgroup $F(\Gamma)$ of Γ were in $\mathscr{C}$, then $|F(\Gamma)| = p$ and $n = |\Gamma| = pq$, for two distinct primes p and q, by **18b**. Hence there would exist two prime subplanes of orders p, q in $\mathbf{P}$, and as n cannot be a square, **3.2.18** would imply that $p^2 + p \leqq n = pq \geqq q^2 + q$, which leads to a contradiction. Hence $F(\Gamma) \notin \mathscr{C}$, and **18b** gives $|F\Gamma| = p^m$, so that $n = |\Gamma| = p^m q$, again with p and q distinct primes. The groups $\Lambda_L(u,x)$, with $c \neq x \mathrm{I} uv = A$, commute in pairs, because of (11); by a result of Wielandt [1951, Satz 8], the same is true for their Fitting subgroups $\Phi_L(u,x) = F(\Lambda_L(u,x))$. Hence the product of all the $\Phi_L(u,x)$ is a p-group Σ of permutations of $(L) - \{c\}$, and all Σ-orbits have length $\geqq p^m$. As $p^{m+1} \nmid n$, some Σ-orbit $\mathfrak{c} \leqq (L) - \{c\}$ must have exact length p^m, and $\mathfrak{c}$ is then a $\Phi_L(u,x)$-orbit for every $x \neq c$ on $A = uv$. Let s, t be two points in $\mathfrak{c}$ and $x \neq c, u$ on A. Then the subplane $\langle s, t, u, x \rangle$ must be of order p, and there exists $\lambda \in \Lambda_A(s,t)$, of order $o(\lambda) = p$, such that $u^\lambda = x$. This means that a Sylow p-group of $\Lambda_A(s,t)$ must be transitive on the points $\neq c$ of A, and now the regularity of $\Lambda_A(s,t)$ implies $n = p^m$, a contradiction. This shows that $\mathscr{C}$ cannot be a Frobenius partition.

Now assume that **19** is false, i.e. that n has two distinct prime divisors. Then **18c** shows that Γ must be simple, for the existence of a proper normal subgroup would imply $|\Gamma| = n = p^2$. Hence we can use **18e** and conclude from known properties of $PGL_2(q)$, $PSL_2(q)$ [Dickson 1901, p. 285) and $Sz(q)$ [Suzuki 1962a, Section 13] that the only remaining possibility is $n = 60$ and $\Gamma \cong PSL_2(4) \cong A_5$, the alternating group of degree 5. In order to exclude this also, consider besides $\Gamma = \Lambda_L(u,v)$ the groups $\Delta = \Lambda_L(u,w)$ and $\Phi = \Lambda_L(v,w)$. All these groups are isomorphic to A_5, and if Σ denotes the group generated by Γ and Δ, then

$$\Sigma = \Gamma\Delta = \Delta\Gamma \supset \Phi \tag{12}$$

because of **15b** and **16**, so that in particular $|\Sigma| > 60$. Now we use the following fact:

20. *A group of order* > 60 *which is the product* $AB = BA$ *of two copies* A, B *of* A_5 *is isomorphic to either* $A \times B$ *or* A_6.

Proof: Kegel & Lüneburg 1963, Satz B. In view of this, it will now be sufficient to prove that Σ cannot be either of A_6 or $A_5 \times A_5$.

Assume first $\Sigma \cong A_6$. Then there are precisely two conjugate classes of subgroups isomorphic to A_5, and two subgroups in the same class have intersection of order $\geqq 12$ [Dickson 1901, Chapter 12]. But from (12) and $|\Sigma| = 360 = |A_5|^2/10$ it follows that $|\Gamma \cap \Delta| = |\Delta \cap \Phi| = |\Phi \cap \Gamma| = 10$, so that we have the desired contradiction.

Finally assume $\Sigma \cong A_5 \times A_5$. Then we consider the *diagonals* of Σ, i.e. the subgroups isomorphic to $\{(x, x^\alpha) : x \in A_5\}$, with $\alpha \in \mathrm{Aut} A_5$. Any subgroup of $A_5 \times A_5$ which is not a direct factor but isomorphic to A_5 must be such a diagonal. As every automorphism of A_5 fixes some element $\neq 1$, two distinct diagonals must have intersection $\neq 1$. But under our present assumption we have

$$|\Gamma \cap \Delta| = |\Delta \cap \Phi| = |\Phi \cap \Gamma| = 1,$$

so that we can assume without loss in generality that $\Sigma = \Gamma \times \Phi$ and $\Delta = \Delta_\alpha = \{\gamma\gamma^\alpha : \gamma \in \Gamma\}$, with α an isomorphism from Γ onto Φ. But then a stabilizer Σ_x of any point $x \neq c$ on L has order 60 and trivial intersection with Γ and Φ; hence it must be a diagonal $\Delta_\beta = \{\gamma\gamma^\beta : \gamma \in \Gamma\}$. But the above remark implies that then $\Delta \cap \Sigma_x \neq 1$, and this contradicts the regularity of Δ on $(L) - \{c\}$. This completes the proof of **19**.

We can now give the improvement of **14** mentioned above.

21. *If the little Reidemeister condition is satisfied in a finite projective plane* **P**, *then* **P** *is desarguesian.*

In view of **19**, it suffices to prove this under the additional assumption that the order n of **P** is a prime power p^e. The following argument is due to Gleason 1956, Theorem 2.5. Let Π denote the permutation group generated by all the groups $\Lambda_L(u, x)$ of order n, with u a fixed point $\neq c$, and x ranging over all points $\neq u, c$ of $A = uc$. By (11), the p-groups $\Lambda_L(u, x)$ commute pairwise; thus Π as the product of these groups, is also a p-group. Consequently, Π has a nontrivial centre. As Π contains the set Λ of **15a**, again by (11), it follows from **15a** that there exist nontrivial (c, A)-elations in **P**. This is true for every flag (c, A) of **P**, and an appeal to **2.3.27b** finally shows that **P** is desarguesian.[1]

---

[1] The case $\lambda = 1$ was actually excluded in **2.3.27**; it was only mentioned there that **2.3.27b** holds also if $\lambda = 1$ (see **4.3.22a** below). The proof of **21** may be finished without **2.3.27b** as follows (Gleason 1956, Lemma 1.6): Let Σ denote the group generated by all elations of **P**. As $\Sigma(c, A) \neq 1$ for every flag (c, A), result **3.1.14** shows that the subgroups $\Sigma(x, A)$ of $\Sigma(A, A)$ all have the same order $h > 1$, for any line A; hence $\Sigma(A, A)$ has order

$$1 + (h-1)(n+1).$$

Also, as $\Sigma(A, A)$ is regular on the n^2 points not on A, the order of $\Sigma(A, A)$ divides n^2. Hence $n^2 = m[1 + (h-1)(n+1)]$ for some integer $m > 0$. But this equation shows (i) that $m \equiv 1 \bmod n + 1$ and (ii) that $m < n$, because $h > 1$. Hence $m = 1$; this means $|\Sigma(A, A)| = n^2$ or (A, A)-transitivity of **P**. As A was an arbitrary line, result **3.122h** (and its context) show that **P** is desarguesian.

We note a fairly immediate consequence of **21**. If condition (5), with the additional restriction that $q_2 \mathrm{I}\, p_1 p_4$,[1] holds in a projective plane **P**, then every ternary field of **P** has a commutative additive loop. But by a result of Bol 1938 and Baer 1939, these loops are then also associative [see also Pickert 1955, p. 49], and this is equivalent to the little Reidemeister condition in **P**. Thus (5) with $q_2 \mathrm{I}\, p_1 p_4$ implies this condition. Combining this and the dual argument with **7**, **14**, **21**, and **3.1.22i**, we can summarize our results as follows:

22. *The following properties of a finite projective plane* **P** *are equivalent:*

(a) **P** *is desarguesian.*
(b) **P** *is (L, L)-transitive for all lines L.*
(c) **P** *is (L, L)- and (M, M)-transitive for two distinct lines L and M.*
(d) **P** *satisfies the Reidemeister condition* (9).
(e) **P** *satisfies the little Reidemeister condition, i.e.* (9) *whenever $c \mathrm{I}\, u v$.*
(f) **P** *satisfies condition* (5) *whenever $q_2 \mathrm{I}\, p_1 p_4$.*
(g) **P** *satisfies condition* (5) *whenever $(p_1 p_3)(p_2 p_4) \mathrm{I}\, q_1 q_2$.*
(h) **P** *is pappian, i.e. satisfies* (5) *without restriction.*

A more direct proof that **(f)** implies **(a)** is in Lüneburg 1960.

We conclude this section with some remarks on *planes with characteristic*. Let **P** be a projective plane and $\mathfrak{q} = (o, e, u, v)$ an ordered quadrangle in **P**. Consider the permutation $\lambda = \lambda_{ev}((o\, e)\,(u\, v), u)$ of $(o\, v) - \{v\}$, as defined by (7), and let $\mathfrak{c}$ be the λ-cycle containing o. If $\mathfrak{c}$ is infinite, we say that $\mathfrak{q}$ has *characteristic* 0; clearly this cannot happen in a finite plane. Otherwise, the *characteristic* of $\mathfrak{q}$ is the integer $|\mathfrak{c}|$. Note that this definition depends on the ordering of the four points in $\mathfrak{q}$, and that in the desarguesian case the characteristic of any quadrangle coincides with the characteristic of the coordinatizing field of the plane, hence it is either 0 or a prime number.

A projective plane **P** is said to be of *characteristic* m if every ordered quadrangle in **P** has characteristic m. It is easy to see that if a finite plane of order n has characteristic m, then m is a divisor of n. Some other simple relations between m and n were proved by Lombardo-Radice 1955a.

The only known finite planes with characteristic are the desarguesian ones; here m is a prime number. It has been conjectured that there are no others, but so far this can be proved only in the minimal case:

23. *A finite projective plane of characteristic 2 is desarguesian.*

---

[1] This is also known as the *axial*, and its dual (cf. **22g** below) as the *central, little theorem of Pappus;* see Pickert 1955, p. 153.

Note that a projective plane is of characteristic 2 if and only if each of its quadrangles has collinear diagonal points;[1] cf. Fig. 7. For the proof of **23** (GLEASON 1956, Theorem 3.5), consider the set $\Lambda_L(u, v)$ for arbitrary line L, and with $c \mathrm{I} u\, v$. The requirement of characteristic 2 implies that all permutations $\neq 1$ in $\Lambda_L(u, v)$ are of order 2, and the product of any two distinct permutations in $\Lambda_L(u, v)$ is likewise an involution. It follows that the elements of $\Lambda_L(u, v)$ commute pairwise, so that $\Lambda_L(u, v)$ is an abelian group. But then **13** shows that the little Reidemeister condition is satisfied, and **21** yields **23**.

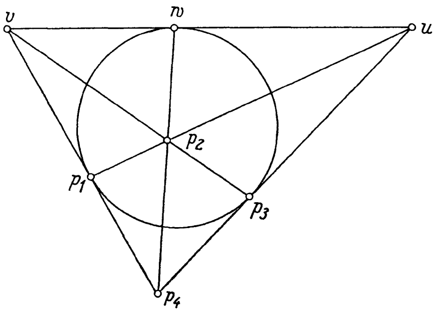

Fig. 7 The plane **P**(2)

Little beyond **23** is known about finite planes with characteristic. Note that **17** implies:

24. *If the little hexagonality condition holds in a finite plane of characteristic m, then m is a prime number, and every quadrangle generates a subplane isomorphic to* $\mathbf{P}(m)$.

This was rediscovered by DEMARIA 1959. LOMBARDO-RADICE 1955a has shown that **24** can be improved in the case of characteristic 3:

25. *A projective plane is of characteristic 3 if and only if every quadrangle generates a subplane isomorphic to* $\mathbf{P}(3)$.

Thus the hexagonality condition is superfluous here. Further results on planes with characteristic may be found in KEEDWELL 1963, 1964.

---

[1] GLEASON 1956 calls such a plane a "Fano plane", despite the fact that the condition known as the Axiom of FANO [1896] forbids the occurence of quadrangles with collinear diagonal points. For this reason, ZADDACH 1956, 1957, calls the same planes "Anti-Fano planes". He shows in these papers that certain infinite planes of characteristic 2 (for example those generated by a quadrangle and a point on one of its sides) are also desarguesian.