More particularly, the geometric meaning of alternative division rings was first studied by Ruth Moufang (see the references in [2] or [6]) and characterized by the uniqueness of the projective construction of a fourth harmonic point. Later Hall [2] gave an independent characterization in terms of his Theorem L. Theorem L is a projective axiom which, in two of its Euclidean forms, becomes* respectively the vector axiom and the distributive axiom. The characterization of Veblen-Wedderburn systems is originally due to Hall.

One final remark. It is an amusing fact that Theorem 6.4 of Hall [2], although true, was not completely proved until 1953, just ten years after the publication date of Hall's paper. By a trivial slip, the theorem contains the word "two" where "three" would have been appropriate. In 1950 I had a rude awakening in this connection which led to Theorem 2. The record shows that a similar experience led Skornyakov to the study of right alternative division rings.—May there be more such slips!

*Appendix I. Planar ternary rings.* A planar ternary ring is a system $(R, F)$ consisting of a set $R$ and a ternary operation $F$ subject to the following postulates:

(i) 0 and 1 are two distinct elements of $R$.

(ii) If $a, b, c$ are in $R$, $F(a, b, c)$ is a uniquely defined element of $R$.

(iii) $F(0, b, c) = F(a, 0, c) = c$ for all $a, b, c$ of $R$.

(iv) $F(a, 1, 0) = F(1, a, 0) = a$ for each $a$ in $R$.

(v) If $b, b', c, c'$ are in $R$, with $b \neq b'$, the equation $F(x, b, c) = F(x, b', c')$ has a unique solution $x$ in $R$.

(vi) If $a, a', b, b'$ are in $R$, with $a \neq a'$, the system of equations $F(a, x, y) = b$, $F(a', x, y) = b'$ has a unique solution $x, y$ in $R$.

(vii) If $a, b, c$ are in $R$, the equation $F(a, b, x) = c$ has a unique solution $x$ in $R$.

A planar ternary ring $(R, F)$ determines a unique Euclidean plane defined as follows: The points of the plane are the ordered pairs $(x, y)$ of elements $x, y$ of $R$. Each ordered pair $[m, b]$ of elements $m, b$ of $R$ is a line of the plane which passes through those points $(x, y)$ such that $y = F(x, m, b)$. Each symbol $[a]$, $a$ in $R$, is a line of the plane which passes through those points $(x, y)$ such that $x = a$.

*Addition* is defined for a planar ternary ring $(R, F)$ by $a+b = F(a, b, 0)$. The system $(R, +)$ is a loop. That is:

(viii) In the equation $x+y=z$, if any two of $x, y, z$ are assigned as elements of $R$, the third is uniquely determined as an element of $R$.

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(*e.g.*, $RS$, $XY$) appear parallel. You should recognize the figure for the vector axiom (triangles $ZXY$, $TRS$). Then note that Theorem L can be interpreted as the vector axiom: $ZY$ is parallel to $TS$. Now begin afresh with Hall's Figure 4 (or, equivalently, restore the deleted line $AMNB$ and its points). This time delete line $ARX$ and its points and apply the same process. Theorem L, as now interpreted, gives a statement about triangles $NZT$, $MYS$ which is slightly different from but clearly equivalent to the distributive axiom.—As an alternative suggestion, the reader may prefer to consult a pamphlet by H. G. Forder [12] which (I am told—I have not yet seen it) covers much the same ground as the present paper with more emphasis on the projective formulation.

* See footnote pp. 15–16.

(ix) There exists an element 0 of $R$ such that $0+a=a+0=a$ for every $a$ in $R$.

A loop $(R, +)$ is a group provided:

(x) $(a+b)+c=a+(b+c)$ for all $a$, $b$, $c$ of $R$, and is an abelian group if also

(xi) $a+b=b+a$ for all $a$, $b$ of $R$.

*Multiplication* is defined for a planar ternary ring $(R, F)$ by $ab=F(a, b, 0)$. In particular,

(xii) $0a=a0=0$ for all $a$ in $R$.

If $R^*$ consists of $R$ with 0 removed, $(R^*, \cdot)$ is a loop; that is, (viii), (ix) hold with $R$, $+$, 0 replaced by $R^*$, $\cdot$, 1 respectively.

*Appendix II. Special planar ternary rings.* A Veblen-Wedderburn system is a system $(R, +, \cdot)$ consisting of a set $R$ and two binary operations $+$, $\cdot$, subject to the following postulates:

(I) $(R, +)$ is an abelian group with zero 0.

(II.1) $(a+b)c=ac+bc$ for all $a$, $b$, $c$ of $R$.

(III) $(R^*, \cdot)$ is a loop with identity 1.

(IV) $a0=0$ for each $a$ of $R$.

(V) If $a$, $a'$, $b$ are in $R$, with $a\neq a'$, the equation $xa=xa'+b$ has a unique solution $x$ in $R$.

A Veblen-Wedderburn system $(R, +, \cdot)$ becomes a planar ternary ring $(R, F)$ when $F$ is defined by $F(a, b, c)=ab+c$.

A division ring (with identity element 1) is a system $(R, +, \cdot)$ which satisfies (I), (II.1), (III) and

(II.2) $c(a+b)=ca+cb$ for all $a$, $b$, $c$ of $R$.

Every division ring (with identity) is a Veblen-Wedderburn system, but not conversely.

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