Math 130
Homework 3 – Due September 20, 2016
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1. Recall that a number is *algebraic* if it is the root of a polynomial in $\mathbb{Q}[x]$. It is *algebraic of degree* n if the lowest-degree polynomial of which it is a root has degree n.

   (a) Prove that if a is algebraic of degree n, then $\sqrt{a}$ is algebraic of degree at most $2n$. Must it be exactly $2n$?

   (b) Prove that if a is algebraic of degree n, then a^2 is algebraic of degree at most n. Must it be exactly n?

   (c) Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2})(\sqrt{3})$, and conclude that this field has degree 4 over $\mathbb{Q}$.

2. Show that:

   (a) if L_1 and L_2 are two line segments whose endpoints have coordinates in a field F, then the intersection of L_1 and L_2 has coordinates in F.

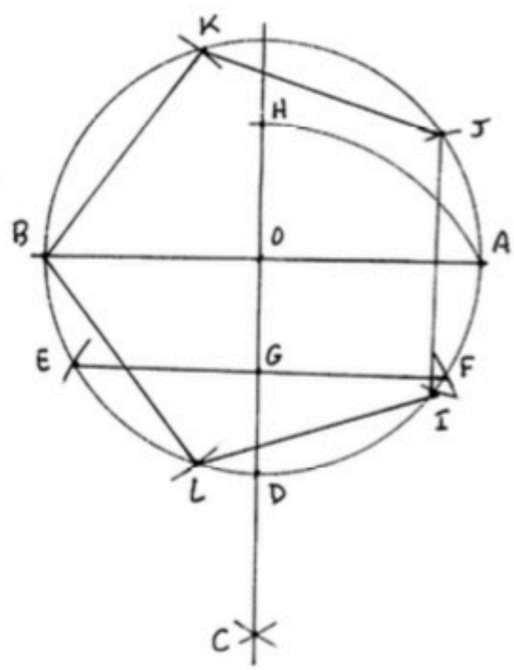
   (b) if $(x-a)^2+(y-b)^2 = r^2$ and $(x-c)^2+(y-d)^2 = s^2$ are two circles, with $a, b, c, d, r, s \in F$, then a point (x, y) where the circles intersect is the root of a degree-2 polynomial with coefficients in F. *Hint: replace one equation with the difference of the two equations to reduce the problem to the intersection of a circle and a straight line.*

3. Prove that the construction below gives a regular pentagon.

1. Draw any line through O. Get A, B.
2. Circle AB.
3. Circle BA, get C.
4. OC, get D.
5. Circle DO. Get E, F.
6. EF, get G.
7. Circle GA, get H.
8. Circle center A, radius OH, get I, J.
9. Circle center B, radius IJ, get K, L.

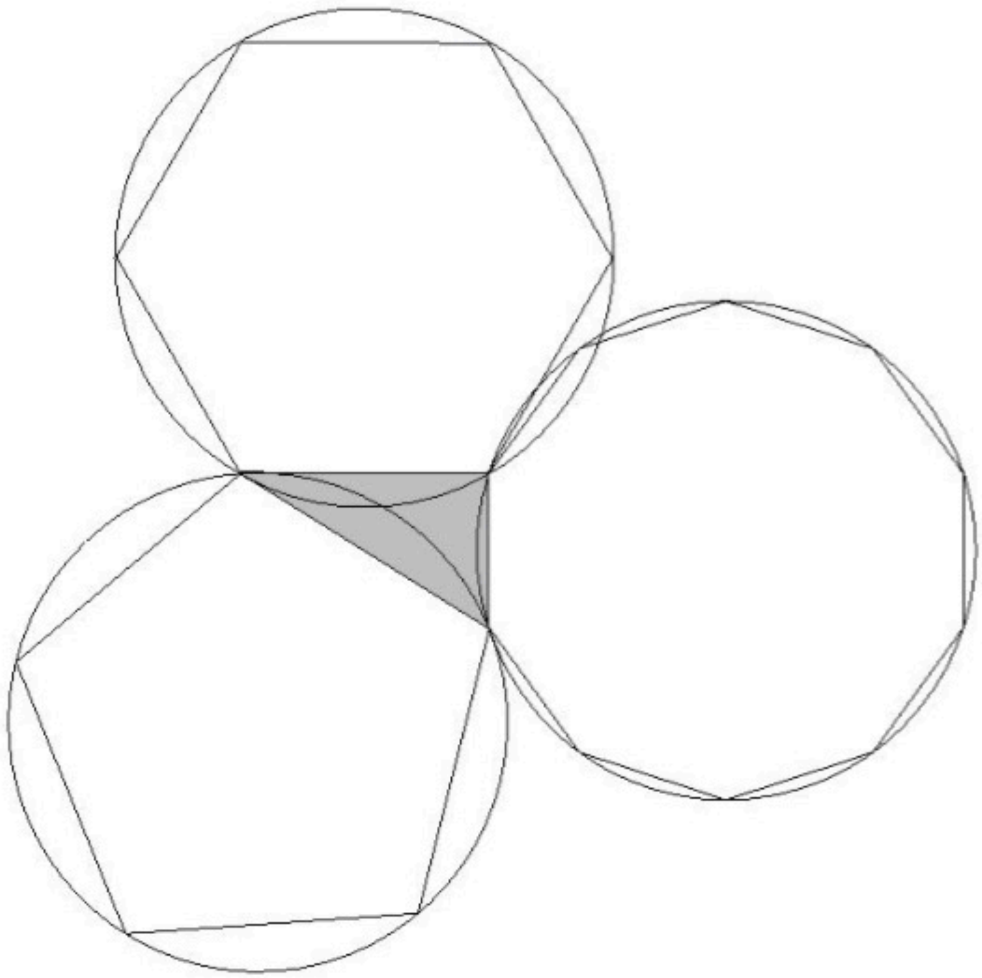
10–14. Draw BK, KJ, JI, IL, LB.

Then $BKJIL$ is the required pentagon.



4. (a) Prove that the triangle formed by the sides of an inscribed hexagon, pentagon, and decagon is right-angled. See the illustration. You may use any high-school-level geometry in your proof.

   (b) Using (a), show that the segment AH in (3) has the same length as any side of the pentagon.



5. (a) Prove that the regular 9-gon is not constructible. You may use results that we proved in class.

   (b) More generally, prove that an angle of d degrees, where $0 < d < 180$ is an integer, is constructible if and only if d is a multiple of 3.

6. Show, without using Gauss' theorem, that if you can construct a regular m-gon and a regular n-gon, where m and n are relatively prime, then you can construct a regular mn-gon.