Math 130
Homework 10 – Due November 22, 2016
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1. Build a paper model of the hyperbolic plane tiled by triangles, such that there are 7 triangles around a vertex (ie. a tiling by equilateral triangles of angles $\frac{2\pi}{7}$, $\frac{2\pi}{7}$, and $\frac{2\pi}{7}$). A template of equilateral triangles can be found on the course website. Now, draw some geodesic (straight lines) on the model. To draw a geodesic in a given direction, start by drawing a straight line inside a triangle (don't aim for the corners!), and when you get to the edge, momentarily flatten the edge and continue the straight line on the other side.

   (a) Draw two geodesics that start off as parallel segments, but eventually diverge.

   (b) Can you draw a (big) rectangle with geodesic sides (*ie.* 4 sides, all angles $\frac{\pi}{2}$)? Drawing it inside a single flat triangle piece is cheating. If not, why not?

   (c) Draw the image of some of your geodesic segments on the Poincaré disc by copying them off of your paper model onto the image on the following page. Can you describe what they look like?

   (d) Estimate the area of a disc of radius r on the hyperbolic plane by counting triangles within a disc of fixed radius, and looking for a pattern. This is a very open-ended question, but I expect that you should be able to make an argument that a disc of radius r (for r large) has area at least as big as $\lambda 2^r$, for some constant λ. Perhaps you can even do better.

   You do not have to hand in your paper model, just your answers to parts (b-d).

2. We saw in class that a torus S_1 (or equivalently, $S_{1,0,0}$) can be represented by a 4-sided polygon with marked sides. Classify the surfaces that correspond to all polygons with 2 or 4 sides. Can you classify the hexagons with marked sides?

3. We saw in class that a choice of Euclidean geometry on the torus S_1 corresponds to a choice of two linearly independent vectors in $\mathbb{R}^2$, and that two Euclidean geometries were *the same* if the pairs of vectors could be identified by rotations. Prove that the set of Euclidean geometries on S_1 with unit area can be identified with the upper-half plane $\mathbb{H}$. *Hint: try to normalize any given pair of vectors to a "standard picture".*

   (Note that this is pretty meta: if the set can be identified with $\mathbb{H}$, then we can make that set a geometry in its own right. Thus, the set of unit-area Euclidean geometries on the torus is itself a hyperbolic plane!)