MAZUR-TYPE MANIFOLDS WITH $L$–SPACE BOUNDARY

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ABSTRACT. In this note, we prove that if the boundary of a Mazur-type 4–manifold is an irreducible Heegaard Floer homology $L$–space, then the manifold must be the 4–ball, and the boundary must be the 3–sphere. We use this to give a new proof of Gabai’s Property R.

1. INTRODUCTION

A Mazur-type manifold is a contractible 4–manifold with a particular handle structure: namely, it consists of a single handle of each index 0, 1, and 2, where the 2–handle is attached along a knot $K$ that intersect the co-core of the 1–handle algebraically once (this yields a trivial fundamental group). Let $M(n)$ denote such a manifold, where $n \in \mathbb{Z}$ denotes the framing of the knot along which the 2–handle is attached. Our main result is that

**Theorem 1.** If $M(n)$ is a Mazur-type manifold, and the boundary is an irreducible Heegaard Floer homology $L$–space, then $M(n)$ is diffeomorphic to $B^4$ and $\partial M(n)$ is diffeomorphic to $S^3$.

![Figure 1](image.png)

**Figure 1.** A Mazur-type manifold, with one 0–handle, one 1–handle, and one 2–handle attached along $K$ with framing $n$.

Recall that a Heegaard Floer homology $L$–space (or simply $L$–space) is a 3–manifold whose Heegaard Floer homology is as simple as possible: $HF^{\text{red}}(M, s)$ vanishes for every $s \in \text{Spin}^c(M)$.

**Remark 2.** Our result above provides further evidence to support Ozsváth and Szabó’s conjecture in [16, page 40] that the full list of irreducible homology spheres that are $L$–spaces up to diffeomorphism is $S^3$ and the Poincaré homology sphere $\Sigma(2, 3, 5)$ with its two orientations. This conjecture has already been verified for Seifert-fibered spaces in [17]. Indeed, if we further assume in Theorem 1 that the boundary is a Seifert-fibered space, then the list of Seifert-fibered $L$–spaces, as just mentioned, is $S^3$ and $\pm \Sigma(2, 3, 5)$. By Rohlin’s theorem, $\Sigma(2, 3, 5)$ cannot bound an acyclic manifold. In particular, the first part of Theorem 1 holds trivially in this case. However, there are abundant examples of Mazur-type manifolds with hyperbolic boundary, including the Mazur corks [1,3].

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Given a handle decomposition of a Mazur-type manifold $W$, we can turn it upside down and consider it as being composed of a single handle of indices 2, 3, and 4. Attaching just the 2–handle, we see that we have a surgery on $-\partial W$ that results in $S^1 \times S^2$. We use this to give another proof of (a slightly more general version of) Property R, first proved by Gabai [6].

**Theorem 3.** If $Y$ is an irreducible integer homology sphere L–space, and 0–surgery on $K \subset Y$ gives $S^1 \times S^2$, then $Y$ is $S^3$ and $K$ is the unknot.

We note that our proof via Heegaard Floer homology and contact geometry is of a different flavor than Gabai’s original proof, although some of the machinery in the background is similar to the machinery involved in existing proofs (by Gabai [6], Gordon and Luecke [9], and Scharlemann [18]). Our methods do not require assuming that $Y$ is $S^3$ to start off; however, the other proofs actually prove much more general results.

**Remark 4.** Recall that a well-known equivalent phrasing of the smooth 4–dimensional Poincaré Conjecture is that every contractible manifold with boundary $S^3$ is diffeomorphic to $B^4$ (see [19, Remark 4.8] and related discussion after Question 1.2 in [13]). Theorem 1 touches on this, in that it shows that whenever $S^3$ bounds a contractible manifold $M$ of Mazur-type, then $M$ is diffeomorphic to $B^4$. However, our methods do not generalize to the case of contractible manifolds with more than a single handle of index 1 and 2: in particular, we rely on a result [2, Proposition 1.2] of Akbulut and Karakurt about Mazur-type manifolds, and its natural generalization to the more general setting is no longer true. Indeed, if the proof of Theorem 1 generalized, then the boundary of the co-core of each 2–handle would have to be an unknot. However, there are examples where this is not the case, see for example [8, Section 6].

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## 2. Proofs of Results

**Proof of Theorem 1.** We split our proof into two steps: we first show that the boundary is $S^3$, and then we show that the 4–manifold itself is $B^4$.

We start by recalling that the Heegaard Floer homology of $\partial M(n)$ is independent of the framing $n$. As we mentioned in the introduction, this was proved by Akbulut and Karakurt in [2, Proposition 1.2]. The idea is as follows: since $M(n)$ is contractible, its boundary is an integral homology sphere, and hence

$$HF^+(\partial M(k)) \cong \mathcal{T}^+ \oplus HF_{\text{red}}(\partial M(n)),$$

where $\mathcal{T}^+ \cong \mathbb{F}[U, U^{-1}]/(U \cdot \mathbb{F}[U])$. Therefore, they just need to show that $HF_{\text{red}}(\partial M(n))$ is independent of $n$. This is achieved by applying the Heegaard Floer surgery exact triangle. Namely, $1$– and 0–surgeries along the knot $K'$ produces $M(n+1)$ and $S^1 \times S^2$, respectively, and this fits into the following surgery exact triangle:

$$\cdots \xrightarrow{f_3} HF_k^+(\partial M(n+1)) \xrightarrow{f_2} HF_{k-\frac{1}{2}}^+(S^1 \times S^2, t_0) \xrightarrow{f_1} HF_{k-1}^+(\partial M(n)) \xrightarrow{f_0} \cdots$$
Here, $HF^+(S^1 \times S^2, t_0) \cong T^+_n \oplus T^-_{-n}$, and the homomorphisms $f_1$ and $f_2$ are homogenous of degree $-\frac{1}{2}$. Using these facts, one can quickly determine that $f_3$ induces an isomorphism between $HF_{\text{red}}(\partial M(n))$ and $HF_{\text{red}}(\partial M(n+1))$ (see [2, Proposition 1.2] for more details). Applying Akbulut and Karakurt’s result shows that if $\partial M(n)$ is an $L$–space for one value of $n$, then it is an $L$–space for all values of $n$.

**Step 1:** Assume that $Y = \partial M(n)$ is an $L$–space for some $n$. We want to show that $Y$ is diffeomorphic to $S^3$. Let $K'$ denote a meridian of $K$ (see Figure 2). Thought of as a knot in $Y$, $K'$ is isotopic to the boundary of the co-core of the $2$–handle. Note that $\pm 1$–surgery on $K' \subset Y$ is an $L$–space, since it gives us the $3$–manifolds $\partial M(n \mp 1)$, which are $L$–spaces, by the previous paragraph.

![Figure 2. The knot $K' \subset Y$.](image)

If $Y$ is not diffeomorphic to $S^3$, then we claim that the complement of $K'$ in $Y$ is irreducible. This can be seen as follows: since $Y$ is itself irreducible, if $K'$ has reducible complement, then $K'$ must be contained in a $3$–ball. If this is the case, then the result of $0$–surgery on $K'$ would be the connected sum $Y \# Y'$, for some $3$–manifold $Y'$. However, we know that the result of $0$–surgery on $K'$ is actually $S^1 \times S^2$, since $K'$ is the meridian of $K \subset S^1 \times S^2$. Since $S^1 \times S^2$ is prime, it follows that $K'$ must have an irreducible complement.

Since $K'$ is an $L$–space knot, and either $Y \cong S^3$ or the complement of $K'$ is irreducible, then by [4] Theorem 6.5] (see also [12, Page 1, paragraph 2]), it follows that $K'$ must be fibered. On the other hand, by [11] and [5, Corollary 1.4], fibered $L$–space knots support tight contact structures. This is proved by calculating the Heegaard Floer contact invariant of a certain contact structure on $-(Y_n(K'))$, where $n \in \mathbb{Z}$ is large. If $K'$ supports a contact structure with vanishing Heegaard Floer contact invariant, then one shows that the reduced Heegaard Floer contact invariant for the contact structure $-(Y_n(K'))$ is non-vanishing, which cannot happen if it is an $L$–space.

However, both $K'$ in $Y$ and $-K'$ in $-Y$ (its mirror) are fibered $L$–space knots, since both $1$– and $-1$–surgery on $K'$ yields an $L$–space. If they both support tight contact structures, then the monodromy of the compatible open book must be trivial. Since $Y$ is a homology sphere, this implies that the page of the open book is a disk, that $K'$ is the unknot, and that $Y$ is diffeomorphic to $S^3$.

**Step 2:** We now wish to show that $M(n)$ is diffeomorphic to $B^4$. First recall that if $M(n)$ admits a Stein structure in which $\partial M(n)$ is a convex level-set of the plurisubharmonic function, then $M(n)$ is a Stein filling — and hence a strong symplectic filling — of the tight contact structure on $S^3$. By a famous result of Gromov and McDuff [10,14], any minimal such strong symplectic filling is diffeomorphic to $B^4$. 

Let $k$ be a positive integer, such that $M(n - k)$ admits a Stein structure. To find such a $k$, let $L \subset (S^1 \times S^2, \xi_{std})$ be a Legendrian realization of $K$, the attaching sphere of the 2-handle. We can now measure $tb(L)$ (see [7, Section 2] for details and conventions), such that we can build a Stein structure on $M(tb(L) - 1)$ by extending the Stein structure on $S^1 \times B^3$ over a Stein 2-handle attached to $L$ with smooth framing $tb(L) - 1$. Now, we can choose any $k$ such that $n - k \leq tb(L) - 1$.

![Figure 3](image.png)

Now note that $S^3$ can be described as the boundary of $M(n - k)$ with 2-handles attached with framing $-1$ along $k$ copies of $K' \subset S^3 = \partial M(n - k)$. As a 4-manifold,

$$M(n - k) \cup 2\text{-}\text{handles} \cong M(n - k)\#k\mathbb{CP}^2,$$

and since $M(n - k)$ admits a Stein structure, this $k$–fold blow-up admits a symplectic structure with strongly convex boundary (see [15, Section 7.1]). Additionally, by pulling off the attaching spheres of the 2-handles off of $K$, the attaching sphere of the 2-handle of $M(n - k)$ (see Figure 3), we see that this manifold also describes $M(n)\#k\mathbb{CP}^2$, and hence the latter manifold also admits a symplectic structure with strongly convex boundary. By blowing down, we find that $M(n)$ itself admits a symplectic structure with strongly convex boundary (see again [15, Section 7.1]). Since $M(n)$ is minimal, the aforementioned result of Gromov and McDuff implies that $M(n)$ is diffeomorphic to $B^4$.

**Proof of Theorem 3.** Let $Y$ be an irreducible integer homology sphere $L$–space, and let $K' \subset Y$ be a knot such that 0–surgery on $K'$ gives $S^1 \times S^2$. Consider the 4–dimensional cobordism from $Y$ to $S^1 \times S^2$ that is the trace of this surgery. Turn this cobordism upside down, to see it as a cobordism from $S^1 \times S^2$ to $-Y$, and glue on $S^1 \times B^3$ by a diffeomorphism $S^1 \times S^2 \cong \partial(S^1 \times B^3)$. Call the resulting 4–manifold $W$, and notice that $W$ is a Mazur-type manifold, and $K'$ is isotopic to boundary of the co-core of the 2-handle in $W$. By Theorem 1 and its proof, we know that $-Y \cong S^3$ (and hence $Y \cong S^3$ as well), and also that $K'$ is the unknot.

**Remark 5.** Given Property R, showing that any Mazur-type manifold with boundary $S^3$ is actually diffeomorphic to $B^4$ (Step 2 in our proof of Theorem 1) is trivial: turning it upside down, it must consist of a 2–handle attached along an unknot and a canceling 3–handle, followed by a capping 4–handle, which gives $B^4$. However, we find that the symplectic geometric proof presents an unusual take on this problem that we find interesting.
REFERENCES


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