MAZUR-TYPE MANIFOLDS WITH L–SPACE BOUNDARY

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ABSTRACT. In this note, we prove that if the boundary of a Mazur-type 4–manifold is an irreducible Heegaard Floer homology L–space, then the manifold must be the 4–ball, and the boundary must be the 3–sphere. We use this to give a new proof of Gabai's Property R.

1. INTRODUCTION

A *Mazur-type* manifold is a contractible 4–manifold with a particular handle structure: namely, it consist of a single handle of each index 0, 1, and 2, where the 2–handle is attached along a knot K that intersect the co-core of the 1–handle algebraically once (this yields a trivial fundamental group). Let $M(n)$ denote such a manifold, where $n \in \mathbb{Z}$ denotes the framing of the knot along which the 2–handle is attached. Our main result is that

Theorem 1. *If $M(n)$ is a Mazur-type manifold, and the boundary is an irreducible Heegaard Floer homology L–space, then $M(n)$ is diffeomorphic to B^4 and $\partial M(n)$ is diffeomorphic to S^3.*

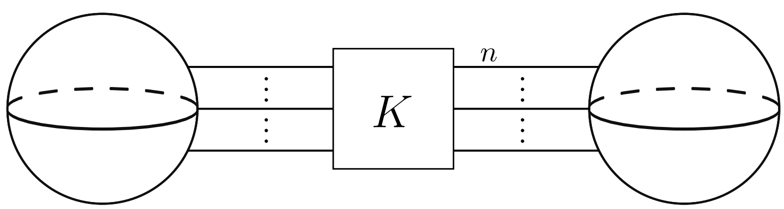

FIGURE 1. A Mazur-type manifold, with one 0–handle, one 1–handle, and one 2–handle attached along K with framing n.

Recall that a *Heegaard Floer homology L–space* (or simply *L–space*) is a 3–manifold whose Heegaard Floer homology is as simple as possible: $HF^{\mathrm{red}}(M, \mathfrak{s})$ vanishes for every $\mathfrak{s} \in \mathrm{Spin}^c(M)$.

Remark 2. Our result above provides further evidence to support Ozsváth and Szabó's conjecture in [16, page 40] that the full list of irreducible homology spheres that are L–spaces up to diffeomorphism is S^3 and the Poincaré homology sphere $\Sigma(2, 3, 5)$ with its two orientations. This conjecture has already been verified for Seifert-fibered spaces in [17]. Indeed, if we further assume in Theorem 1 that the boundary is a Seifert-fibered space, then the list of Seifert-fibered L–spaces, as just mentioned, is S^3 and $\pm\Sigma(2, 3, 5)$. By Rohlin's theorem, $\Sigma(2, 3, 5)$ cannot bound an acyclic manifold. In particular, the first part of Theorem 1 holds trivially in this case. However, there are abundant examples of Mazur-type manifolds with hyperbolic boundary, including the Mazur corks [1,3].

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Given a handle decomposition of a Mazur-type manifold W, we can turn it upside down and consider it as being composed of a single handle of indices 2, 3, and 4. Attaching just the 2–handle, we see that we have a surgery on $-\partial W$ that results in $S^1 \times S^2$. We use this to give another proof of (a slightly more general version of) Property R, first proved by Gabai [6].

Theorem 3. *If Y is an irreducible integer homology sphere L–space, and 0–surgery on $K \subset Y$ gives $S^1 \times S^2$, then Y is S^3 and K is the unknot.*

We note that our proof via Heegaard Floer homology and contact geometry is of a different flavor than Gabai's original proof, although some of the machinery in the background is similar to the machinery involved in existing proofs (by Gabai [6], Gordon and Luecke [9], and Scharlemann [18]). Our methods do not require assuming that Y is S^3 to start off; however, the other proofs actually prove much more general results.

Remark 4. Recall that a well-known equivalent phrasing of the smooth 4–dimensional Poincaré Conjecture is that every contractible manifold with boundary S^3 is diffeomorphic to B^4 (see [19, Remark 4.8] and related discussion after Question 1.2 in [13]). Theorem 1 touches on this, in that it shows that whenever S^3 bounds a contractible manifold M of Mazur-type, then M is diffeomorphic to B^4. However, our methods do not generalize to the case of contractible manifolds with more than a single handle of index 1 and 2: in particular, we rely on a result [2, Proposition 1.2] of Akbulut and Karakurt about Mazur-type manifolds, and its natural generalization to the more general setting is no longer true. Indeed, if the proof of Theorem 1 generalized, then the boundary of the co-core of each 2–handle would have to be an unknot. However, there are examples where this is not the case, see for example [8, Section 6].

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2. Proofs of Results

Proof of Theorem 1. We split our proof into two steps: we first show that the boundary is S^3, and then we show that the 4–manifold itself is B^4.

We start by recalling that the Heegaard Floer homology of $\partial M(n)$ is independent of the framing n. As we mentioned in the introduction, this was proved by Akbulut and Karakurt in [2, Proposition 1.2]. The idea is as follows: since $M(n)$ is contractible, its boundary is an integral homology sphere, and hence

$$HF^+(\partial M(k)) \cong \mathcal{T}^+ \oplus HF^{\mathrm{red}}(\partial M(n)),$$

where $\mathcal{T}^+ \cong \mathbb{F}[U, U^{-1}]/(U \cdot \mathbb{F}[U])$. Therefore, they just need to show that $HF^{\mathrm{red}}(\partial M(n))$ is independent of n. This is achieved by applying the Heegaard Floer surgery exact triangle. Namely, -1– and 0–surgeries along the knot K' produces $M(n+1)$ and $S^1 \times S^2$, respectively, and this fits into the following surgery exact triangle:

$$\cdots \xrightarrow{f_3} HF^+_k(\partial M(n+1)) \xrightarrow{f_1} HF^+_{k-\frac{1}{2}}(S^1 \times S^2, \mathfrak{t}_0) \xrightarrow{f_2} HF^+_{k-1}(\partial M(n)) \xrightarrow{f_3} \cdots$$

Here, $HF^+(S^1 \times S^2, \mathfrak{t}_0) \cong \mathcal{T}^+_{\frac{1}{2}} \oplus \mathcal{T}^+_{-\frac{1}{2}}$, and the homomorphisms f_1 and f_2 are homogenous of degree $-\frac{1}{2}$. Using these facts, one can quickly determine that f_3 induces an isomorphism between $HF^{\mathrm{red}}(\partial M(n))$ and $HF^{\mathrm{red}}(\partial M(n+1))$ (see [2, Proposition 1.2] for more details). Applying Akbulut and Karakurt's result shows that if $\partial M(n)$ is an L–space for one value of n, then it is an L–space for all values of n.

Step 1: Assume that $Y = \partial M(n)$ is an L–space for some n. We want to show that Y is diffeomorphic to S^3. Let K' denote a meridian of K (see Figure 2). Thought of as a knot in Y, K' is isotopic to the boundary of the co-core of the 2–handle. Note that ± 1–surgery on $K' \subset Y$ is an L–space, since it gives us the 3–manifolds $\partial M(n \mp 1)$, which are L–spaces, by the previous paragraph.

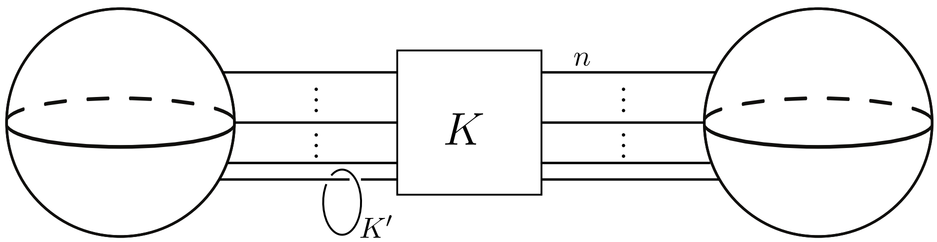

FIGURE 2. The knot $K' \subset Y$.

If Y is not diffeomorphic to S^3, then we claim that the complement of K' in Y is irreducible. This can be seen as follows: since Y is itself irreducible, if K' has reducible complement, then K' must be contained in a 3–ball. If this is the case, then the result of 0–surgery on K' would be the connected sum $Y \# Y'$, for some 3–manifold Y'. However, we know that the result of 0–surgery on K' is actually $S^1 \times S^2$, since K' is the meridian of $K \subset S^1 \times S^2$. Since $S^1 \times S^2$ is prime, it follows that K' must have an irreducible complement.

Since K' is an L–space knot, and either $Y \cong S^3$ or the complement of K' is irreducible, then by [4, Theorem 6.5] (see also [12, Page 1, paragraph 2]), it follows that K' must be fibered. On the other hand, by [11] and [5, Corollary 1.4], fibered L–space knots support tight contact structures. This is proved by calculating the Heegaard Floer contact invariant of a certain contact structure on $-(Y_n(K'))$, where $n \in \mathbb{Z}$ is large. If K' supports a contact structure with vanishing Heegaard Floer contact invariant, then one shows that the reduced Heegaard Floer contact invariant for the contact structure $-(Y_n(K'))$ is non-vanishing, which cannot happen if it is an L–space.

However, both K' in Y and $-K'$ in $-Y$ (its mirror) are fibered L–space knots, since both 1– and -1–surgery on K' yields an L–space. If they both support tight contact structures, then the monodromy of the compatible open book must be trivial. Since Y is a homology sphere, this implies that the page of the open book is a disk, that K' is the unknot, and that Y is diffeomorphic to S^3.

Step 2: We now wish to show that $M(n)$ is diffeomorphic to B^4. First recall that if $M(n)$ admits a Stein structure in which $\partial M(n)$ is a convex level-set of the plurisubharmonic function, then $M(n)$ is a Stein filling — and hence a strong symplectic filling — of the tight contact structure on S^3. By a famous result of Gromov and McDuff [10, 14], any minimal such strong symplectic filling is diffeomorphic to B^4.

Let k be a positive integer, such that $M(n-k)$ admits a Stein structure. To find such a k, let $L \subset (S^1 \times S^2, \xi_{\mathrm{std}})$ be a Legendrian realization of K, the attaching sphere of the 2–handle. We can now measure $tb(L)$ (see [7, Section 2] for details and conventions), such that we can build a Stein structure on $M(tb(L)-1)$ by extending the Stein structure on $S^1 \times B^3$ over a Stein 2–handle attached to L with smooth framing $tb(L)-1$. Now, we can choose any k such that $n-k \leq tb(L)-1$.

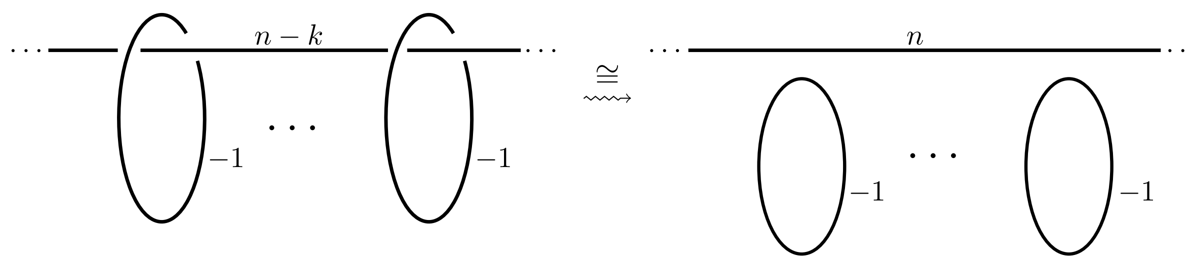

FIGURE 3.

Now note that S^3 can be described as the boundary of $M(n-k)$ with 2–handles attached with framing -1 along k copies of $K' \subset S^3 = \partial M(n-k)$. As a 4–manifold,

$$M(n-k) \bigcup 2\text{–handles} \cong M(n-k) \# k\overline{\mathbb{CP}}^2,$$

and since $M(n-k)$ admits a Stein structure, this k–fold blow-up admits a symplectic structure with strongly convex boundary (see [15, Section 7.1]). Additionally, by pulling off the attaching spheres of the 2–handles off of K, the attaching sphere of the 2–handle of $M(n-k)$ (see Figure 3), we see that this manifold also describes $M(n)\#k\overline{\mathbb{CP}}^2$, and hence the latter manifold also admits a symplectic structure with strongly convex boundary. By blowing down, we find that $M(n)$ itself admits a symplectic structure with strongly convex boundary (see again [15, Section 7.1]). Since $M(n)$ is minimal, the aforementioned result of Gromov and McDuff implies that $M(n)$ is diffeomorphic to B^4. $\square$

Proof of Theorem 3. Let Y be an irreducible integer homology sphere L–space, and let $K' \subset Y$ be a knot such that 0–surgery on K' gives $S^1 \times S^2$. Consider the 4–dimensional cobordism from Y to $S^1 \times S^2$ that is the trace of this surgery. Turn this cobordism upside down, to see it as a cobordism from $S^1 \times S^2$ to $-Y$, and glue on $S^1 \times B^3$ by a diffeomorphism $S^1 \times S^2 \cong \partial(S^1 \times B^3)$. Call the resulting 4–manifold W, and notice that W is a Mazur-type manifold, and K' is isotopic to boundary of the co-core of the 2–handle in W. By Theorem 1 and its proof, we know that $-Y \cong S^3$ (and hence $Y \cong S^3$ as well), and also that K' is the unknot. $\square$

Remark 5. Given Property R, showing that any Mazur-type manifold with boundary S^3 is actually diffeomorphic to B^4 (Step 2 in our proof of Theorem 1) is trivial: turning it upside down, it must consist of a 2–handle attached along an unknot and a canceling 3–handle, followed by a capping 4–handle, which gives B^4. However, we find that the symplectic geometric proof presents an unusual take on this problem that we find interesting.

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