TIGHT CONTACT STRUCTURES VIA ADMISSIBLE TRANSVERSE SURGERY

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ABSTRACT. We investigate the line between tight and overtwisted for surgeries on fibred transverse knots in contact 3-manifolds. When the contact structure ξ_K is supported by the fibred knot $K \subset M$, we obtain a characterisation of when negative surgeries result in a contact structure with non-vanishing Heegaard Floer contact class. To do this, we leverage information about the contact structure $\xi_{\overline{K}}$ supported by the mirror knot $\overline{K} \subset -M$. We derive several corollaries about the existence of tight contact structures, L-space knots outside S^3, non-planar contact structures, and non-planar Legendrian knots.

1. INTRODUCTION

Contact geometers have long been fascinated with the subtle art of distinguishing tight contact structures from overtwisted one: the border between these two categories is poorly understood. In this paper, we investigate one manifestation of this problem: when does transverse surgery on transverse knots in overtwisted contact structures result in a tight contact structure?

We focus on fibred knots $K \subset M$, where the fibre surface is a (rational) Seifert surface for K; to such a knot is naturally associated a contact structure ξ_K, turning K into a transverse knot. Both null-homologous fibred knots and rationally fibred knots (*ie.* rationally null-homologous and fibred) support a unique contact structure up to isotopy (see [1, 16]).

What draws our attention to such knots is a result of Etnyre and Vela-Vick ([14]), namely, that ξ_K restricted to the complement of K is always tight, even when (M, ξ_K) is overtwisted. Following the guiding principle that "removing twisting" of the contact planes increases the chances of being tight, we will investigate the operation of *admissible transverse surgery*: this operation defines a contact structure on $M_r(K)$ that has "less twisting" of the contact planes than the original contact structure.

We know that this operation has a hope of succeeding, because when we remove a "sufficient amount" of twisting, it is known that this operation will result in a tight contact structure (*cf.* Corollary 1.3). We will look at what we can say when an arbitrary small amount of twisting is removed. In this setting, the more negative the surgery coefficient, the less twisting is removed.

To make the above more precise, we say a few words about how the topological operation of surgery interacts with the contact geometry. Topological r-surgery on a null-homologous fibred knot K induces a rational fibred knot K_r in $M_r(K)$, for $r \neq 0$. The contact structure ξ_{K_r} on $M_r(K)$ induced by the fibred knot K_r was determined by Baker, Etnyre, and van Horn-Morris (in [1]) (for $r < 0$) and by the author ([7]) (for $r > 0$) to be the contact structure obtained by admissible (resp. inadmissible) transverse r-surgery on K in (M, ξ_K). In the cases we will consider, admissible (resp. inadmissible) transverse surgery on K corresponds to negative (resp. positive) contact surgery on a Legendrian approximation of K (this is not true in general; see [6]).

The main result of this paper is a characterisation of the Heegaard Floer contact invariant of the contact structure ξ_{K_r}, for $r < 0$. In particular, we will be concerned with $c^+(\xi_r(K)) \in HF^+(-M_r(K))$ and its image $c_{red}^+(\xi_{K_r})$ in $HF^+(-M_r(K)) \to HF_{red}(-M_r(K))$, defined by Ozsváth and Szabó in [36]. Among other properties, the non-vanishing of either of these classes implies that the contact structure ξ_{K_r} is tight.

To state the result, we also need to consider the "mirror" of K as a fibred knot in $-M$ (that is, M with reversed orientation). To emphasise the ambient manifold, we will denote this knot as $\overline{K} \subset -M$, and the supported contact structure on $-M$ as $\xi_{\overline{K}}$.

Theorem 1.1. *If $(M_r(K), \xi_{K_r})$ is the result of admissible transverse r-surgery on $K \subset (M, \xi_K)$, then:*

(1) $c^+(\xi_{K_r}) \neq 0$ *for all $r < 0$ if and only if $c^+_{\mathrm{red}}(\xi_{\overline{K}}) = 0$.*
(2) $c^+_{\mathrm{red}}(\xi_{K_r}) \neq 0$ *for all $r < 0$ if and only if $c^+(\xi_{\overline{K}}) = 0$.*

This theorem will proved in slightly more generality (taking surgery rationally fibred knots into account) as Theorem 3.1, and in Corollary 3.2, we look at the particular values of r at which the contact invariants change from zero to non-zero.

Ozsváth and Szabó took the first steps toward this result: in [36], they showed that when $HF_{\mathrm{red}}(M) = 0$, then $c^+(\xi_{K_{-1}}) \neq 0$ for any fibred knot $K \subset M$. Since $HF_{\mathrm{red}}(M) = 0$ implies that $c^+_{\mathrm{red}}(\xi_{\overline{K}}) = 0$, our result extends this to ξ_{K_r}, for all $r < 0$. Similar results appear by Hedden and Plamenevskaya in [19], in the context of L-space surgeries. In [18], Hedden and Mark investigated $c^+(\xi_{K_r})$ for $r = -1/n$, in cases where Theorem 1.1 gives no information. It is always true that $c^+(\xi_{K_{-1/n}}) \neq 0$ for sufficiently large positive integers n (see Corollary 1.3), and they determined upper bounds on the minimum such n (*cf.* Corollary 3.2).

Example 1.2. For an example of how Theorem 1.1 can obviate the need for detailed calculation of Heegaard Floer groups, we show that the open book (Σ, ϕ) in Figure 1 supports a tight contact structure with $c^+_{\mathrm{red}} \neq 0$, for any integers $k_1, k_2 \geqslant 2$. Let K be the binding of this open book, and consider the open book $(\Sigma, \phi^{-1} \circ \tau_\partial)$ for inadmissible transverse (-1)-surgery on $\overline{K}$, where τ_∂ is a boundary parallel Dehn twist. This latter open book is a stabilisation of one considered in [6, Section 4], which was shown to support a contact structure with $c^+ = 0$. By Theorem 1.1, we conclude that (Σ, ϕ) supports a tight contact structure with $c^+_{\mathrm{red}} \neq 0$.
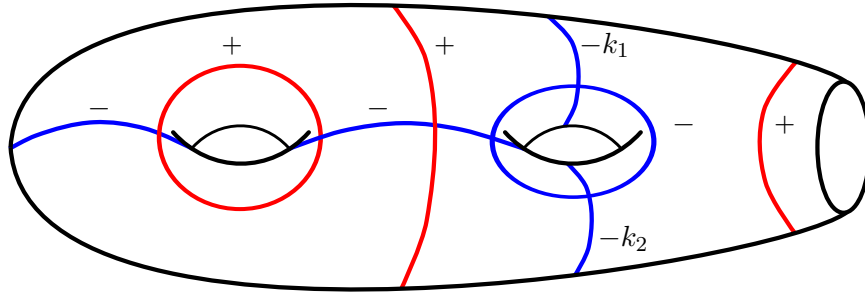

FIGURE 1. A genus-2 open book supporting a tight contact structure with $c^+_{\mathrm{red}} \neq 0$, for any integers $k_1, k_2 \geqslant 2$.

We present several consequences of Theorem 1.1. First, we can recover a familiar result of Honda, Kazez, and Matić. The fractional Dehn twist coefficient is a measure of how much a mapping class "twists" arcs with endpoints on the boundary components of an open book.

Corollary 1.3 ([23]). *Let (Σ, ϕ) be an open book decomposition, where Σ has one boundary component. If the fractional Dehn twist coefficient (FDTC) of ϕ around $\partial\Sigma$ is greater than 1, then the supported contact structure is tight with $c^+_{\mathrm{red}} \neq 0$.*

Their proof involved the construction first of a taut foliation which was then perturbed to a tight contact structure. Our proof does not require passing via taut foliations, nor does it prove their existence (in that sense, we do not recover the full extent of their result).

We call $K \subset M$ an L-space knot if $M_r(K)$ is an L-space for some $r > 0$, ie. $|H_1(M_r(K))| = \mathrm{rank}\, \widehat{HF}(M_r(K))$. In S^3, such knots are known to be fibred ([30]) and support the tight contact

structure ([17]). Outside of S^3, not all L-space knots are fibered. However, when they *are* fibered, we generalise Hedden's result from [17].

Corollary 1.4. *If $K \subset M$ is a fibered L-space knot, then $c^+(\xi_K) \neq 0$.*

The support genus of a contact structure is defined in [13] to be the minimal genus of an open book supporting the contact structure. Ozsváth, Stipsicz, and Szabó showed in [32] that if $c_{\mathrm{red}}^+(\xi) \neq 0$, then ξ is non-planar, *ie.* cannot be supported by an open book with planar pages.

Corollary 1.5. *If $K \subset M$ is fibred, and $c^+(\xi_{\overline{K}}) = 0$, then $(M_r(K), \xi_{K_r})$ is non-planar for all $r < 0$.*

Similarly, consider the support genus of a Legendrian knot $L \subset (M, \xi)$, *ie.* the minimal genus open book supporting ξ such that L sits on a page, defined in [31]. This next corollary generalises a result of Li and Wang ([28]).

Corollary 1.6. *If $K \subset M$ is fibred, and $c^+(\xi_{\overline{K}}) = 0$, and L is a Legendrian approximation of K, then neither L nor $-L$ sit on the page of a planar open book supporting ξ_K, where $-L$ is L with the orientation reversed.*

In particular, no Legendrian approximation of a torus knot (or more generally, any non-trivial fibered strongly quasi-positive knot, by [17]) with maximal Thurston–Bennequin number sits on the page of a planar open book for $(S^3, \xi_{\mathrm{std}})$.

We can use the full generality of Theorem 3.1 to show that sufficiently large inadmissible transverse surgery on a (rationally) null-homologous fibred knot $K \subset (M, \xi_K)$ preserves the non-vanishing of c_{red}^+.

Corollary 1.7. *Let $K \subset M$ be a (rationally) null-homologous fibred knot, and fix a framing of K. If $c_{\mathrm{red}}^+(\xi_K) \neq 0$, then $c_{\mathrm{red}}^+(\xi_{K_r}) \neq 0$ for all sufficiently large r (measured with respect to the fixed framing).*

As an amusing application, Theorem 1.1 gives many examples of rationally fibred knots K such that both K and $\overline{K}$ support tight contact structures. The first such examples were discovered by Hedden and Plamenevskaya in [19, Corollary 2]. This is a phenomenon that for null-homologous fibred knots only occurs when the monodromy of the fibration is isotopic to the identity. To create these examples, start with a null-homologous fibred genus-g knot K' that supports a tight contact structure $\xi_{K'}$ on M', and such that $c_{\mathrm{red}}^+(\xi_{K'}) = 0$. By [7, Theorem 1.6], inadmissible transverse r-surgery on K' is tight for $r > 2g - 1$. Thus, if $K = K'_r$ is the surgery dual knot to K' under this r-surgery, then both K and $\overline{K} = \overline{K'}_{-r}$ support tight contact structures, by [7, Theorem 1.6] and Theorem 1.1, respectively.

Up to this point, we have only considered negative surgery coefficients. This is because admissible transverse surgery is only defined for slopes in an interval $[-\infty, a)$, where a is often hard to pin down. However, when the fibred knot $K \subset M$ supports an overtwisted contact structure ξ_K, and the associated monodromy is not right-veering (see [22] for details), we can show that admissible transverse surgery is defined for any positive surgery coefficient, and that such a surgery is tight.

Theorem 1.8. *Let $K \subset M$ be a null-homologous fibred knot supporting an overtwisted contact structure ξ_K. Assume further that the monodromy ϕ_K of the open book associated to the fibration is not right-veering. Then we can define admissible transverse r-surgery for all $r \in \mathbb{Q}$, and the resulting manifold $(M_r(K), \xi_r)$ satisfies $c_{\mathrm{red}}^+(\xi_r) \neq 0$ for all $r \geqslant 0$ (in addition to whatever is guaranteed by Theorem 1.1).*

Remark 1.9. For $r > 0$, we stress that ξ_r given in Theorem 1.8 is not the contact structure ξ_{K_r} on $M_r(K)$ supported by the rationally fibred knot K_r. Indeed, ξ_r is the result of admissible transverse surgery on K, whereas ξ_{K_r} for $r > 0$ is the result of inadmissible transverse surgery on K (see Theorem 2.8).

We can use Theorems 3.1 and 1.8 to narrow down the possibilities for manifolds that do not support tight contact structures.

Corollary 1.10. *If $HF_{\mathrm{red}}(M) = 0$, and $K \subset M$ is a null-homologous fibred knot, then if $M_r(K)$ does not support a tight contact structure, then ϕ_K is right-veering and $r > 0$.*

For a general M and $K \subset M$ is null-homologous and fibred, there exists at most one $r \in \mathbb{Q} \cup \{\infty\}$ such that neither orientation of $M_r(K)$ supports a tight contact structure. In addition, at least one of the following is true:

- *$M_r(K)$ supports a tight contact structure in both of its orientations for all $r > 0$.*
- *$M_r(K)$ supports a tight contact structure in both of its orientations for all $r < 0$.*

This paper is organised as follows: in Section 2 we recall some Heegaard Floer homology, transverse surgery, and how they interact; in Section 3, we deal with the proof of Theorem 1.1 and Corollaries 1.3–1.7; in Section 4, we prove Theorem 1.8 and Corollary 1.10; and in Section 5, we look at some further questions, namely, the difference between c^+ and $\hat{c}$, and genus-1 fibred knots.

2. BACKGROUND

We assume familiarity with the basics of Heegaard Floer Homology and knot Floer Homology ([34, 35]); Legendrian and transverse knots, surgery on Legendrian knots, open book decompositions, and convex surfaces ([8–11]). Here, we will recall the relevant information.

2.1. **Heegaard Floer Homology.** All of the Heegaard Floer groups in this paper will be defined over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. As per convention, given a generating set $\{\boldsymbol{x}\}$ for $\widehat{CF}(M)$, we consider $CF^{\infty}(M)$ as being generated by $\{U^i\boldsymbol{x} \,|\, i \in \mathbb{Z}\}$. Then, $CF^-(M)$ is generated by $\{U^i\boldsymbol{x} \,|\, i \in \mathbb{Z}_{>0}\}$, and $CF^+(M) = CF^{\infty}(M)/CF^-(M)$. These chain complexes are involved in several short exact sequences, leading to homology long exact sequences.

$$\cdots \xrightarrow{\delta_+} \widehat{HF}(M) \xrightarrow{\hat{\iota}} HF^+(M) \xrightarrow{U} HF^+(M) \xrightarrow{\delta_+} \cdots$$

$$\cdots \xrightarrow{\delta_-} HF^-(M) \xrightarrow{U} HF^-(M) \xrightarrow{\pi_-} \widehat{HF}(M) \xrightarrow{\delta_-} \cdots$$

$$\cdots \xrightarrow{\delta_\infty} HF^-(M) \xrightarrow{\iota_-} HF^\infty(M) \xrightarrow{\pi_\infty} HF^+(M) \xrightarrow{\delta_\infty} \cdots$$

For the second short exact sequence, we naturally identify the chain complex generated by $\{U\boldsymbol{x}\}$ with $\widehat{CF}(M)$. With this identification, the commutativity of the following diagram can be checked on the chain level.

$$
\begin{array}{ccc}
\widehat{HF}(M) & \xrightarrow{\hat{\iota}} & HF^+(M) \\
& {\scriptstyle \delta_-}\searrow \quad \swarrow {\scriptstyle \delta_\infty} & \\
& HF^-(M). &
\end{array}
$$

We define $HF_{\mathrm{red}}(M)$ to be $\operatorname{coker} \pi_\infty$, or equivalently $\ker \iota_-$; these two definitions are isomorphic via δ_∞.

The Heegaard Floer chain groups for $-M$ are dual to those for M. In particular, we get a natural non-degenerate bilinear pairing (the Kronecker pairing) on $\widehat{CF}(-M) \otimes \widehat{CF}(M)$ that descends to homology:

$$\langle \cdot, \cdot \rangle : \widehat{HF}(-M) \otimes \widehat{HF}(M) \to \mathbb{F}.$$

There is a similar pairing for $HF^\pm(-M)$ and $HF^\mp(M)$, see [37] for details. For our purposes, it is sufficient to note that if $\delta_-(x) \neq 0$, where $x \neq 0 \in \widehat{HF}(-M)$ and $\delta_- : \widehat{HF}(-M) \to HF^+(-M)$, then there exists some $y \neq 0 \in \widehat{HF}(M)$ such that $\langle x, y \rangle \neq 0$ and $y \in \operatorname{im} \delta_+$, or equivalently, $\widehat{\imath}(y) = 0 \in HF^+(M)$.

Putting this together with the fact that $\delta_- = \delta_\infty \circ \widehat{\imath}$, we get the following characterisation of elements of $HF_{\mathrm{red}}(-M)$, which we will use in the proof of Theorem 1.1.

Lemma 2.1. *Given an element $x \in \widehat{HF}(-M)$, then $\widehat{\imath}(x) \in HF^+(-M)$ descends to a non-zero element of $HF_{\mathrm{red}}(-M)$ if and only if there exists some $y \neq 0 \in \widehat{HF}(M)$ such that $\langle x, y \rangle \neq 0$ and $\widehat{\imath}(y) = 0 \in HF^+(M)$.*

2.2. **Knot Heegaard Floer Homology.** Ozsváth and Szabó [34] and independently Rasmussen [39] define a filtration on $\widehat{CF}(M)$ induced by a (rationally) null-homologous knot $K \subset M$; this filtration $\mathcal{A}$ is called the *Alexander grading*. We get an induced bi-filtration on $CF^\infty(M)$ given by $\mathcal{F}(U^i \boldsymbol{x}) = (-i, \mathcal{A}(\boldsymbol{x}) - i)$. We will use $\mathcal{C}_K$ to denote the complex $CF^\infty(M)$ filtered by K, and as per convention, $\mathcal{C}_K\{\mathrm{relation}(i,j)\}$ will denote sub and quotient complexes of $\mathcal{C}_K$ determined by $\mathrm{relation}(i,j)$.

Remark 2.2. The Alexander grading depends on a choice of relative Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(M, K)$. However, different choices of $\mathfrak{s}$ give isomorphic chain complexes, and in practice amounts to shifting the Alexander grading by a fixed constant. Since we are only interested in the structure of $\mathcal{C}_K$ and not the particular values of the Alexander grading, we need not worry about this issue. For the remainder of the article, we assume that we have fixed $\mathfrak{s}$, whenever needed. See [19, 33, 34, 38] for more details.

First note that $\mathcal{C}_K\{i = 0\} = \widehat{CF}(M)$, and $\mathcal{C}_K\{i \geqslant 0\} = CF^+(M)$. The homology of quotient complexes of $\mathcal{C}_K\{i = 0\}$ tell us whether K is fibred.

Theorem 2.3 ([30], [19]). *A rationally null-homologous knot $K \subset M$ with irreducible complement is fibred if and only if $H_*(\mathcal{C}_K\{i = 0, j = \mathrm{bottom}\}) \cong \mathbb{F}$, where bottom is the smallest value of j such that the homology is non-trivial. If K is integrally null-homologous, then $\mathrm{bottom} = g(K)$, where $g(K)$ is the genus of the knot.*

Let K be a rationally null-homologous knot with a fixed framing. If K is null-homologous, then we can choose the Seifert framing, and if not, then we can choose the *canonical framing* (see [29, 38]), but this choice will not affect our results.

According to [33, 34], we can use $\mathcal{C}_K$ to determine the Heegaard Floer homology of sufficiently large (positive or negative) surgeries on K (with respect to the fixed framing). For our purposes, it will be sufficient to identify the complexes

$$\mathcal{C}_{K_n}\{i = 0, j = \mathrm{bottom}\}, \mathcal{C}_{K_n}\{i = 0\}, \text{ and } \mathcal{C}_{K_n}\{i \geqslant 0\},$$

where K_n is the surgery dual knot to K after n-surgery, for a large integer $n > 0$. We will also determine

$$\mathcal{C}_{K_{-n}}\{i = 0, j = \mathrm{top}\}, \mathcal{C}_{K_{-n}}\{i = 0\}, \text{ and } \mathcal{C}_{K_{-n}}\{i \geqslant 0\},$$

where K_{-n} is the surgery dual knot to K after $-n$-surgery, for a large integer $n > 0$.

Let top be the maximum value of j_0 such that $H_*(\mathcal{C}_K\{i = 0, j = j_0\})$ is non-trivial, and define bottom as the minimum such value. Let $\widehat{A} = \mathcal{C}_K\{\max(i, j - \mathrm{top} + 1) = 0\}$, let $A^+ = \mathcal{C}_K\{\max(i, j - \mathrm{top} + 1) \geqslant 0\}$, and let $S = \mathcal{C}_K\{i < 0, j = \mathrm{top} - 1\} = \mathcal{C}_K\{i = -1, j = \mathrm{top} - 1\}$. The last equality holds because for a fixed i_0, we can assume that $\mathcal{C}_K\{i = i_0\}$ is non-trivial only for $\mathrm{bottom} < j - i_0 < \mathrm{top}$.

Theorem 2.4 ([19,33,34]). *Let $K \subset M$ be a rationally null-homologous knot with a fixed framing. For all sufficiently large integers n, we have:*

- $\mathcal{C}_{K_n}\{i = 0, j = \text{bottom}\} \simeq S$,
- $\mathcal{C}_{K_n}\{i = 0\} = \widehat{CF}(M_n(K)) \simeq \widehat{A}$,
- $\mathcal{C}_{K_n}\{i \geqslant 0\} = CF^+(M_n(K)) \simeq A^+$,
- $\widehat{c}(\xi_{K_n})$ *is the image of $H_*(S) \to H_*(A)$.*

Let $\widehat{A}' = \mathcal{C}_K\{\min(i, j - \text{bottom} - 1) = 0\}$, let $A'^+ = \mathcal{C}_K\{\min(i, j - \text{bottom} - 1) \geqslant 0\}$, and let $Q' = \mathcal{C}_K\{i > 0, j = \text{bottom} + 1\} = \mathcal{C}_K\{i = 1, j = \text{bottom} + 1\}$.

Theorem 2.5 ([19,33,34]). *Let $K \subset M$ be a rationally null-homologous knot with a fixed framing. For all sufficiently large integers n, we have:*

- $\mathcal{C}_{K_{-n}}\{i = 0, j = \text{top}\} \simeq Q'$,
- $\mathcal{C}_{K_{-n}}\{i = 0\} = \widehat{CF}(M_{-n}(K)) \simeq \widehat{A}'$,
- $\mathcal{C}_{K_{-n}}\{i \geqslant 0\} = CF^+(M_{-n}(K)) \simeq A'^+$.

Remark 2.6. The proof of [19, Theorem 4.2], on which the first bullet points of both the theorems above are based, goes through with no changes for rationally null-homologous fibred knots, after taking care when calculating the values of the Alexander filtration.

Based on Theorem 2.3, we conclude that if K is fibred, $H_*(Q')$ and $H_*(S)$ are singly generated.

2.3. **Transverse Surgery and Open Books.** Consider a contact manifold (M, ξ) containing a transverse knot K. Fix a framing on K, against which all slopes will be measured. There is a natural analogue of Dehn surgery on transverse knots, called *transverse surgery*. This comes in two flavours: *admissible* and *inadmissible*. See [6,7] for more details.

Given a transverse knot $K \subset (M, \xi)$, a neighbourhood of K is contactomorphic to $\{r \leqslant a\} \subset (S^1 \times \mathbb{R}^2, \xi_{\text{rot}} = \ker(\cos r\, dz + r \sin r\, d\theta))$ for some a, where z is the coordinate on S^1, and K is identified with the z-axis. The slope of the leaves of the foliation induced by the contact planes on the torus $\{r = r_0\}$ is $-\cot r_0/r_0$. We will abuse notation and refer to the torus $\{r = r_0\}$ (resp. the solid torus $\{r \leqslant r_0\}$) as being a torus (resp. a solid torus) of slope $-\cot r_0/r_0$.

If L is a Legendrian approximation of K, then inside a standard neighbourhood of L, we can find a neighbourhood of K where the boundary torus is of slope p/q, for any $p/q < tb(L)$. Conversely, if we have a neighbourhood for K where the boundary torus is of slope p/q, then there exist Legendrian approximations L_n of K with $tb(L_n) = n$, for any integer $n < p/q$.

To perform *admissible transverse surgery*, we take a torus of rational slope p/q inside $\{r \leqslant a\}$, we remove the interior of the corresponding solid torus of slope p/q, and perform a contact cut on the boundary, *ie.* we quotient the boundary by the S^1-action of translation along the leaves of slope p/q (see [27] for details); the contact structure descends to the quotient manifold to give the result of admissible transverse p/q-surgery on K. To perform *inadmissible transverse surgery*, we first remove $\{r < b\}$ for some $b \leqslant a$. We then glue on the $T^2 \times I$ layer $\{r_0 \leqslant r \leqslant b + 2\pi\}$ by identifying $\{r = b + 2\pi\}$ with $\partial (M \backslash \{r < b\})$; we choose r_0 such that the new boundary $\{r = r_0\}$ is a torus of slope p/q. The result of inadmissible transverse p/q-surgery on K is the result of performing a contact cut on the new boundary.

A (rationally) fibred link $L \subset M$ induces an open book decomposition of M, which supports a unique contact structure ξ_L and in which L is a transverse link ([1,16]). The construction of ξ_L gives a natural neighbourhood of each component of L in which we can define admissible transverse surgery up to the slope given by the page (the *page slope*).

Remark 2.7. As in [7, Remark 2.9], the result of admissible transverse surgery depends in general on the neighbourhood of K chosen to define it. For fibred knots $K \subset (M, \xi_K)$, we will always be using the neighbourhood mentioned above, unless otherwise noted.

If a link $L \subset M$ is (rationally) fibred, then topological r-surgery on a component K of L for any r not equal to the page slope gives a rationally fibred knot $L' \subset M'$.

Theorem 2.8 ([1], [7]). *The contact structure $\xi_{L'}$ on M' is the result of admissible (resp. inadmissible) transverse r-surgery on the knot $K \subset (M, \xi_L)$, when r is less than (resp. greater than) the page slope.*

In [7, Section 3.3], the author gave an algorithm to exhibit an abstract integral open book supporting $\xi_{L'}$. To do this, we break up an admissible transverse surgery into a sequence of integral admissible transverse surgeries, and an inadmissible transverse surgery is treated as an inadmissible transverse $(f + 1/n)$-surgery followed by an admissible transverse surgery (where f is the page slope). The algorithm can be described in terms of the following two moves: (1) adding a boundary parallel Dehn twist about a boundary component of the current open book, and (2) stabilising the open book along a boundary parallel arc. The first move is an integral admissible (resp. inadmissible) transverse surgery if we add a positive (resp. negative) Dehn twist, and the second move changes the surgery coefficient of subsequent surgeries. See [7, Section 3.3] for the details.

2.4. **Heegaard Floer Contact Elements.** As in [19,36], when $K \subset M$ is rationally fibred, the *contact element* $\widehat{c}(\xi_K) \in \widehat{HF}(-M)$ of the supported contact structure is given by the image of

$$H_*(\mathcal{C}_{\overline{K}}\{i = 0, j = \text{bottom}\}) \to H_*(\mathcal{C}_{\overline{K}}\{i = 0\}) \cong \widehat{HF}(-M),$$

where $\overline{K}$ is the knot K considered as a subset of $-M$. We denote by $c^+(\xi_K)$ and $c^+_{\text{red}}(\xi_K)$ the elements $\widehat{\iota}(\widehat{c}(\xi_K)) \in HF^+(-M)$ and $\delta_\infty(c^+(\xi_K)) \in HF_{\text{red}}(-M)$. By Lemma 2.1, $c^+_{\text{red}}(\xi_K) \neq 0$ if and only if there exists some $y \neq 0 \in \widehat{HF}(M)$ such that $\langle \widehat{c}(\xi_K), y \rangle \neq 0$ and $\widehat{\iota}(y) = 0 \in HF^+(M)$.

The contact elements interest us because their non-vanishing proves that the contact structure is tight (see [36]). Furthermore, they behave well under capping off binding components of open books ([5]) and admissible transverse surgery, as in the following lemma.
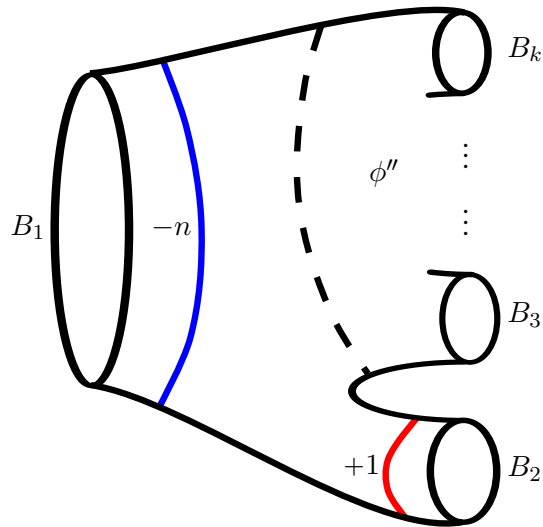

FIGURE 2. This open book is glued along B_1 to do transverse inadmissible s-surgery, for $1/n < s < 1/(n-1)$. The monodromy is $\tau_{B_1}^{-n} \tau_{B_2} \phi''$, where the support of ϕ'' is to the right of the dashed curve.

Lemma 2.9. *Let $K \subset (M, \xi)$ be a transverse knot (not necessarily fibred or null-homologous). If $c^+(\xi) \neq 0$ (resp. $c^+_{\text{red}}(\xi) \neq 0$), then the result (M', ξ') of admissible transverse surgery on K satisfies $c^+(\xi') \neq 0$ (resp. $c^+_{\text{red}}(\xi') \neq 0$).*

Proof. If $K' \subset (M', \xi')$ is the surgery dual knot to K, then some inadmissible transverse s-surgery on K' results in (M, ξ), where s is measured with respect to some framing of K'. By [1, Lemma 6.5], there exists an open book decomposition for (M', ξ') such that K' is a component of the binding. We then stabilise this open book along arcs parallel to the boundary component corresponding to K' to get an open book (Σ', ϕ'), such that $s > 0$ when measured against the framing induced by Σ' on K'. By Theorem 2.8, we can construct an integral open book for (M, ξ) from (Σ', ϕ'), using the algorithm from [7, Section 3.3].

Let 0 be the page slope on K'. If $s = 1/n$ for some positive integer n, then (M, ξ) is supported by the open book $(\Sigma', \phi' \circ \tau_{B'}^{-n})$, where $\tau_{B'}$ is a Dehn twist about the boundary component B' corresponding to K'. By [1, 4], the property that a monodromy supports a contact structure with non-vanishing contact invariant is preserved under composition. Hence, since $(\Sigma', \tau_{B'}^n)$ supports a contact structure with non-vanishing contact invariant, and $\phi' = (\phi' \circ \tau_{B'}^{-n}) \circ \tau_{B'}^n$, so too does (Σ', ϕ') support a contact structure with non-vanishing contact invariant.

If $1/n < s < 1/(n-1)$ (where if $n = 1$, we mean just $s > 1$), then by [7, Section 3.3], (M, ξ) is supported by an open book $(\Sigma' \cup \Sigma'', \phi' \circ \tau_{B_1}^{-n} \tau_{B_2} \phi'')$. Here, the open book $(\Sigma'', \tau_{B_1}^{-n} \tau_{B_2} \phi'')$ has boundary components $B_1, B_2, \ldots, B_k$ (where $k \geqslant 3$), and ϕ'' is supported in a neighbourhood of $B_3 \cup \cdots \cup B_k$ (see Figure 2); we glue Σ' to Σ'' by identifying B_1 with B' (the binding component corresponding to K'). In this open book for (M, ξ), if we cap off $B_3, \ldots, B_k$, we arrive at an open book $(\Sigma', \phi' \circ \tau_{B'}^{-n+1})$.

Since by [5], capping off an open book preserves the non-vanishing of the contact class, if $n = 1$, then we are done. If $n > 1$, then $-n + 1 > 0$, and we have an open book for inadmissible transverse $1/(-n + 1)$-surgery on K', where the supported contact structure has non-vanishing contact invariant. We finish the proof as in the case $s = 1/(-n + 1)$. □

3. NEGATIVE SURGERIES

In this section, we will state and prove Theorem 1.1 in full generality (as Theorem 3.1), and then prove its corollaries.

Theorem 3.1. *Let $K \subset M$ be a rationally null-homologous fibred knot with a fixed framing, and let f be the slope on $\partial N(K)$ induced by a page of the fibration. $(M_r(K), \xi_{K_r})$ is the result of admissible transverse r-surgery on $K \subset (M, \xi_K)$, where r is measured with respect to the fixed framing, then:*

(1) $c^+(\xi_{K_r}) \neq 0$ *for all $r < f$ if and only if $c_{\mathrm{red}}^+(\xi_{\overline{K}}) = 0$.*
(2) $c_{\mathrm{red}}^+(\xi_{K_r}) \neq 0$ *for all $r < f$ if and only if $c^+(\xi_{\overline{K}}) = 0$.*

Proof. To keep notation simple, we will assume K is null-homologous, where the fixed framing is taken to be the Seifert framing (or equivalently, the page slope f), and we fix a relative Spin^c structure to measure the Alexander grading on $\mathcal{C}_K$ and $\mathcal{C}_{\overline{K}}$ such that $\mathrm{top} = g$ and $\mathrm{bottom} = -g$, where $g = g(K)$ is the genus of K. The proof for the more general case is the same, *mutatis mutandis*.

It follows from Lemma 2.9 that the vanishing of $c^+(\xi_{K_r})$ (resp. $c_{\mathrm{red}}^+(\xi_{K_r})$) for some $r < 0$ implies the same vanishing for ξ_{K_n}, where n is a sufficiently negative integer.

Thus, to prove Theorem 3.1 we only need to show that the theorem is true when r is restricted to be a sufficiently negative integer n, using Theorems 2.4 and 2.5.

Proof of (1). Since $c^+(\xi_{K_n}) \in HF^+(-(M_n(K))) = HF^+((-M)_{-n}(\overline{K}))$, we will look at $\mathcal{C}_{\overline{K}}$, and large positive integer surgeries on $\overline{K}$. We start by noting two short exact sequences (where the left map is inclusion), which commute with the obvious vertical inclusions:

$$0 \longrightarrow \mathcal{C}_{\overline{K}}\{i = 0, j \leqslant g - 1\} \longrightarrow \mathcal{C}_{\overline{K}}\{i = 0\} \longrightarrow \mathcal{C}_{\overline{K}}\{i = 0, j = g\} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{C}_{\overline{K}}\{i = 0, j \leqslant g - 1\} \longrightarrow \mathcal{C}_{\overline{K}}\{i \geqslant 0\} \longrightarrow \mathcal{C}_{\overline{K}}\{\max(i, j - g + 1) \geqslant 1\} \longrightarrow 0$$

Denote the left-hand complexes by D. We know that $\mathcal{C}_{\overline{K}}\{i = 0\} = \widehat{CF}(-M)$ and $\mathcal{C}_{\overline{K}}\{i \geqslant 0\} = \widehat{CF}(-M)$. The two right-hand complexes can be identified via U with $\mathcal{C}_{\overline{K}}\{i = -1, j = g - 1\} = S$ and $\mathcal{C}_{\overline{K}}\{\max(i, j - g + 1) \geqslant 0\} = A^+$. The following commutative diagram comes from the homology long exact sequence associated to the above diagram.

$$
\begin{array}{ccccccc}
H_*(D) & \xrightarrow{\widehat{\iota}_D} & \widehat{HF}(-M) & \xrightarrow{\widehat{\pi}} & H_*(S) & \xrightarrow{\widehat{\partial}} & H_*(D) \\
\Big\downarrow{\cong} & & \Big\downarrow{\widehat{\iota}} & & \Big\downarrow{\iota_S} & & \Big\downarrow{\cong} \\
H_*(D) & \xrightarrow{\iota_D^+} & HF^+(-M) & \xrightarrow{\pi^+} & H_*(A^+) & \xrightarrow{\partial^+} & H_*(D)
\end{array}
$$

By Theorem 2.3, $H_*(S)$ is generated by a single element x. The quotient complex $\mathcal{C}_{\overline{K}}\{i = 0, j = g\}$ is dual to the subcomplex $\mathcal{C}_K\{i = 0, j = -g\}$, and $\widehat{c}(\xi_{\overline{K}})$ is the image of

$$H_*(\mathcal{C}_K\{i = 0, j = -g\}) \to \widehat{HF}(M).$$

It follows that if $\widehat{c}(\xi_{\overline{K}}) \neq 0$, then any element $y \in \widehat{\pi}^{-1}(x) \subset \widehat{HF}(-M)$ will pair non-trivially with $\widehat{c}(\xi_{\overline{K}})$. Thus, based on Theorem 2.4 and the discussion in Section 2.4, we deduce the following facts:

- $c_{\mathrm{red}}^+(\xi_{\overline{K}}) \neq 0$ if and only if $\widehat{\partial}(x) = 0$ and there is a $y \in \widehat{\pi}^{-1}(x)$ such that $\widehat{\iota}(y) = 0$;
- $c^+(\xi_{K_n}) = 0$ for sufficiently negative integers n if and only if $\iota_S(x) = 0$.

First assume that $c_{\mathrm{red}}^+(\xi_{\overline{K}}) \neq 0$. As above, we can find some non-zero $y \in \widehat{HF}(-M)$ such that $\widehat{\pi}(y) = x$ and $\widehat{\iota}(y) = 0$. But then by commutativity, $\iota_S(x) = \pi^+\widehat{\iota}(y) = 0$, and so $c^+(\xi_{K_n}) = 0$.

Now assume that $c^+(\xi_{K_n}) = 0$. Thus, $\iota_S(x) = 0$. Since $\widehat{\partial}(x) = \partial^+\iota_S(x) = 0$, we know that $\widehat{c}(\xi_{\overline{K}}) \neq 0$. Then, pick a non-zero element $y_0 \in \widehat{\pi}^{-1}(x)$. If $\widehat{\iota}(y_0) = 0$, then we're done, so assume that $\widehat{\iota}(y_0) \neq 0$. Since $\pi^+(\widehat{\iota}(y_0)) = \iota_S(x) = 0$, we can find some element $d \in H_*(D)$ such that $\iota_D^+(d) = \widehat{\iota}(y_0)$. But now, $y_0 - \widehat{\iota}_D(d) \in \widehat{\pi}^{-1}(x)$, and $\widehat{\iota}(y_0 - \widehat{\iota}_D(d)) = \widehat{\iota}(y_0) - \iota_D^+(d) = 0$. Thus, $c_{\mathrm{red}}^+(\xi_{\overline{K}}) \neq 0$.

Proof of (2). This time, we will consider $\mathcal{C}_K$, and sufficiently negative integer surgeries on K. As in (1), we start by noting two short exact sequences (where the left map is inclusion), which commute with the obvious vertical inclusions:

$$0 \longrightarrow \mathcal{C}_K\{i = 0, j \geqslant -g + 1\} \longrightarrow \mathcal{C}_K\{\max(i, j + g - 1) = 0\} \longrightarrow \mathcal{C}_K\{i \geqslant 1, j = -g + 1\} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{C}_K\{i = 0, j \geqslant -g + 1\} \longrightarrow \mathcal{C}_K\{\max(i, j + g - 1) \geqslant 0\} \longrightarrow \mathcal{C}_K\{i \geqslant 1\} \longrightarrow 0$$

Denote the left-hand complexes by D. The two middle complexes are $\widehat{A}'$ and A'^+, and the top-right complex is Q' (recall, $\mathcal{C}_K\{i > 1, j = -g + 1\}$ is empty). The two right-hand complexes can also

be identified via U with $\mathcal{C}_K\{i = 0, j = -g\}$ and $CF^+(M)$. The following commutative diagram comes from the homology long exact sequence associated to the above diagram.

$$
\begin{array}{ccccccc}
H_*(D) & \xrightarrow{\ \widehat{\iota}_D\ } & H_*(\widehat{A}') & \xrightarrow{\ \widehat{\pi}\ } & H_*(Q') & \xrightarrow{\ \widehat{\partial}\ } & H_*(D) \\
\Big\downarrow{\cong} & & \Big\downarrow{\iota_A} & & \Big\downarrow{\iota_Q} & & \Big\downarrow{\cong} \\
H_*(D) & \xrightarrow{\ \iota_D^+\ } & H_*(A'^+) & \xrightarrow{\ \pi^+\ } & HF^+(M) & \xrightarrow{\ \partial^+\ } & H_*(D)
\end{array}
$$

By Theorem 2.3, $H_*(Q')$ is generated by a single element x. The complex $Q' = \mathcal{C}_{K_n}\{i = 0, j = \text{top}\}$ is dual to $S = \mathcal{C}_{\overline{K}_{-n}}\{i = 0, j = \text{bottom}\}$, and so if $\widehat{c}(\xi_{K_n}) \neq 0$, then any element $y \in \widehat{\pi}^{-1}(x) \subset H_*(\widehat{A}')$ will pair non-trivially with $\widehat{c}(\xi_{K_n})$. Also, since Q' can be identified via U with $\mathcal{C}_K\{i = 0, j = -g\}$, the map ι_Q can be identified with the map

$$
H_*(\mathcal{C}\{i = 0, j = -g\}) \to \widehat{HF}(M) \to HF^+(M).
$$

Thus, based on Theorem 2.5 and the discussion in Section 2.4, we deduce the following facts:

- $c_{\text{red}}^+(\xi_{\overline{K}_n}) \neq 0$ for sufficiently negative integers n if and only if $\widehat{\partial}(x) = 0$ and there is a $y \in \widehat{\pi}^{-1}(x)$ such that $\iota_A(y) = 0$;
- $c^+(\xi_{\overline{K}}) = 0$ if and only if $\iota_Q(x) = 0$.

The rest of the proof follows exactly as in the proof of (1) above. $\qquad \square$

Proof of Corollary 1.3. If $K \subset M$ is the fibred knot giving the open book decomposition (Σ, ϕ), then the open book decomposition associated to $\overline{K}$ is (Σ, ϕ^{-1}), and if $FDTC(\phi) > 1$, then $FDTC(\phi^{-1}) < -1$. By [1], the contact structure $\xi_{\overline{K}_{-1}}$ is supported by $(\Sigma, \phi^{-1} \circ \tau_\partial)$, where τ_∂ is a positive Dehn twist around the binding of Σ. By [26], $\text{FDTC}(\phi^{-1} \circ \tau_\partial) = \text{FDTC}(\phi^{-1}) + 1 < 0$, and thus $\xi_{\overline{K}_{-1}}$ is overtwisted, by [22]. Since $c^+(\xi_{\overline{K}_{-1}}) = 0$, the corollary now follows from Theorem 3.1. $\qquad \square$

Proof of Corollary 1.4. If $M_r(K)$ is an L-space for some $r > 0$, then $(-M)_{-r}(\overline{K})$ is also an L-space for the same r. This means that $c_{\text{red}}^+(\xi_{\overline{K}_{-r}})$ must vanish, and so we conclude that $c^+(\xi_K) \neq 0$, by Theorem 3.1. $\qquad \square$

Proof of Corollary 1.5. Since $c^+(\xi_{\overline{K}}) = 0$, Theorem 3.1 implies that $c_{\text{red}}^+(\xi_{K_r}) \neq 0$. Thus, $(M_r(K), \xi_{K_r})$ cannot be planar, by [32]. $\qquad \square$

Proof of Corollary 1.6. If L is a Legendrian approximation of K, then some negative contact surgery on L is equivalent to some admissible transverse surgery on K, by [6]. We know from [31, Theorem 5.10] that if L sits on the page of a planar open book for ξ_K, then the result of negative contact surgery on L is also planar, which contradicts Corollary 1.5. Since after putting $-L$ on the page of an open book, we can reverse the orientation to get L, we see that $-L$ is also not planar. $\qquad \square$

Proof of Corollary 1.7. By Theorem 3.1, $c^+(\xi_{\overline{K}_{-r}}) = 0$ for sufficiently large r, say for $r > N$. Then, for any $s > N$, we claim that $c^+(\xi_{K_s}) \neq 0$. Indeed, if it were not the case, then by Theorem 3.1, sufficiently negative admissible transverse surgery on $\overline{K}_{-s}$ would have non-vanishing c^+ invariant. However, sufficiently negative admissible transverse surgery on $\overline{K}_{-s}$ gives us $\xi_{\overline{K}_{-s'}}$, for some $N < s' < s$, which we know to have vanishing c^+ invariant. $\qquad \square$

Let $K \subset (M, \xi_K)$ be an integrally null-homologous fibred knot, where $c^+(\xi_K) \neq 0$. Let

$$
R_{\text{red}}^+(K) = \inf \left\{ r \in \mathbb{Q} \,\middle|\, c_{\text{red}}^+(\xi_{K_r}) \neq 0 \right\},
$$

and let

$$R^+(K) = \inf \left\{ r \in \mathbb{Q} \,\middle|\, c^+(\xi_{K_r}) \neq 0 \right\}.$$

If $c^+_{\mathrm{red}}(\xi_K) = 0$, then we define $R^+_{\mathrm{red}}(K) = \infty$. Then, the same approach as the proof of Corollary 1.7 will give the following.

Corollary 3.2. *Let $K \subset M$ be an integrally null-homologous fibred knot, such that $c^+(\xi_{\overline{K}}) \neq 0$. Then:*

- $c^+(\xi_{K_r}) = 0$ *for all* $-\infty \leqslant r \leqslant -R^+_{\mathrm{red}}(\overline{K})$.
- $c^+(\xi_{K_r}) \neq 0$ *and* $c^+_{\mathrm{red}}(\xi_{K_r}) = 0$ *for all* $-R^+_{\mathrm{red}}(\overline{K}) < r \leqslant -R^+(\overline{K})$.
- $c^+_{\mathrm{red}}(\xi_{K-r}) \neq 0$ *for all* $-R^+(\overline{K}) < r < 0$.

4. POSITIVE SURGERIES

We have thus far restricted our attention to negative surgeries, as in general, the existence of admissible transverse r-surgeries on a fibred transverse knot $K \subset (M, \xi_K)$ are only guaranteed for r less than the page slope. In fact, if U is the unknot in (S^3, ξ_U), then we know that we cannot do admissible transverse r-surgery for any $r \geqslant 0$. In general, however, if we can thicken the standard neighbourhood of K, we can define admissible transverse r-surgery for some $r \geqslant 0$.

Recall from Remark 2.7 that the result of admissible transverse surgery may depend on the chosen neighourhood, and so is not in general well-defined. Thus far, we have chosen a neighourhood of K that was compatible with the open book decomposition, such that Theorem 2.8 applied. For positive surgeries, there is no such choice. In particular, if K_r is the surgery dual knot to K in $M_r(K)$, then the contact structure coming from admissible transverse r-surgery on $K \subset (M, \xi_K)$ bears no relation to ξ_{K_r} (see Remark 1.9).

For the rest of this section, let $K \subset M$ be a null-homologous transverse knot. If K has a Legendrian approximation L_n with $tb(L_n) = n$, then for any $r < n$, we can find a standard neighbourhood of K inside a standard neighbourhood of L_n which allows admissible transverse r-surgery for any $r < n$.

We will focus on the case where ξ_K is overtwisted, as we can say the most in that situation. Etnyre and van Horn-Morris proved in [12] that the complement of the fibred transverse knot $K \subset (M, \xi_K)$ is non-loose, *ie.* the restriction of ξ_K to $M \backslash K$ is tight. Thus K intersects every overtwisted disc in (M, ξ_K). Following [2], we define the *depth* $d(K)$ of a transverse knot K to be the minimum of $\left| K \cap D \right|$ over all overtwisted discs D. Since K is non-loose, $d(K) \geqslant 1$. The depth of a transverse link, a Legendrian knot, and a Legendrian link can be similarly defined.

We first show that we can pass from transverse knots to Legendrian knots without increasing the depth.

Lemma 4.1. *If K is a transverse knot such that $d(K) = 1$, then there exists a Legendrian approximation L of K such that $d(L) = 1$.*

Proof. We consider a neighbourhood of $\{r = \pi/4, \theta = 0\} \subset (S^1 \times \mathbb{R}^2, \xi_{\mathrm{rot}} = \ker(\cos r \, dz + r \sin r \, d\theta))$ as a generic model for a neighbourhood of K (where $z \sim z + 1$), and let the overtwisted disc $D = \{z = 0, r \leqslant \pi\}$. It would seem preferable to consider $\{r = 0\}$, but since that knot passes through the unique elliptic point of the characteristic foliation on D, it would not represent a generic model.

The contact planes away from $r = 0$ are spanned by $\frac{\partial}{\partial r}$ and the vector $X(r) = -\tan r \frac{\partial}{\partial z} + \frac{\partial}{\partial \theta}$. We define a Legendrian approximation L_{-n} of K' by

$$L_n(t) = \begin{pmatrix} \alpha \, z(t) \\ \pi/4 + \epsilon \cos(nt) \\ -\alpha f(nt) \end{pmatrix},$$

in the coordinates (z, r, θ), for some positive real number α, where

$$z'(t) = -\frac{nf'(nt)}{\tan\left(\pi/4 + \epsilon\cos(nt)\right)}, \; z(0) = 0.$$

We choose $f : \mathbb{R} \to \mathbb{R}$ to be a smooth function such that

$$f(t) = \begin{cases} \frac{\sin t}{m} & t \in \left(0, \frac{\pi}{2} - \delta\right) \cup \left(\frac{3\pi}{2} + \delta, 2\pi\right), \\ m\sin t & t \in \left(\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta\right), \end{cases}$$

where m is a positive integer and $\delta > 0$ is small. We define f to be periodic with period 2π, and we define f on $\left(\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta\right) \cup \left(\frac{3\pi}{2} - \delta, \frac{3\pi}{2} + \delta\right)$ such that f is smooth.

Then, given any $\epsilon > 0$, choose n to be a sufficiently large positive integer such that $z(2\pi) > 1$ and $\max |f|/z(2\pi) < \arctan\left(4\epsilon/\pi\right)$; choose m to be a sufficiently large positive integer such that there exists some value z_0 of z such that $\left|L_n \cap \{z = z_0\}\right| = 1$. If we now let α be $1/z(2\pi)$, then L_{-n} will be a closed Legendrian in an ϵ-neighbourhood of K'. Note that $tb(L_{-n}) = -n$, and $\left|L_{-n} \cap \{z = z_0/z(2\pi), r \leqslant \pi\}\right| = 1$. Thus, $d(L_{-n}) \leqslant 1$.

To show that $d(L_{-n}) \neq 0$, note that a transverse push-off of L_{-n} can be made arbitrarily close to L_{-n}. Thus, if there exists an overtwisted disc in (M, ξ) that is disjoint from L_{-n}, it is also disjoint from a sufficiently small neighbourhood of L_{-n}. Then, the tranverse push-off K of L_{-n} would also be disjoint from this overtwisted disc. However, $d(K) = 1$, and so this cannot happen. □

The next step is to show that if $d(L) = 1$, then L can be negatively destabilised.

Lemma 4.2. *Let $K \subset (M, \xi)$ be a non-loose transverse knot, and let L_n be a Legendrian approximation of K with $tb(L_n) = n$. If $d(K) = d(L_n) = 1$, then L_n can be negatively destabilised to L_m (hence L_m is also a Legendrian approximation of K) for any $n \geqslant m$, where $tb(L_m) = m$ and $d(L_m) = 1$.*

Proof. Let T be the convex boundary of a regular neighbourhood of L_n. Let γ be a curve on T isotopic to a meridian of L_n. Using the Legendrian Realisation Principle ([20, Theorem 3.7]), we can isotope T to another convex torus T' such that γ is Legendrian with $tb(\gamma) = -1$.

Given an overtwisted disc D such that $\left|L_n \cap D\right| = 1$, isotope D such that its intersection with T' is exactly γ. Let $A \subset D$ be the annulus with boundary $\partial D \cup \gamma$. Since the twisting of the contact planes along the boundary components of A is non-positive with respect to the framing given by A, we can perturb A to be convex. Consider the dividing curves Γ on A: since $tb(\gamma) = -1$ and $tb(\partial D) = 0$, Γ intersects γ twice and ∂D zero times. Since L_n is a non-loose Legendrian knot (as K is a non-loose transverse knot), the dividing curves Γ can have no contractible components, by Giroux's Criterion ([15]), so, Γ is an arc parallel to γ, and gives a bypass.

If T'' is the convex torus we get from the bypass A on T', the dividing curves on T'' are meridional. The region in between T' and T'' is a basic slice, and we can find convex tori with dividing curves of slope m for any $n \leqslant m$. We define L_m to be a Legendrian divide of a convex torus with dividing curves of slope m.

In order to determine whether L_m is a positive or negative destabilisation of L_n, consider the convex torus T''', the boundary of a regular neighbourhood of L_{n-1}, the negative stabilisation of L_n (we pick a negative stabilisation since it is non-loose). If we glue the basic slice from T''' to T' to the basic slice from T' to T''', we get a $T^2 \times I$ with a tight contact structure, and convex boundaries with dividing curves of slope $n - 1$ and ∞, *ie.* another basic slice. By the classification in [20] of tight contact structures on basic slices, the signs of the basic slices that we glue together must agree. Thus, since L_{n-1} is a negative stabilisation of L_n, so too is L_n an $(m - n)$-fold negative stabilisation of L_m.

To show that $d(L_m) \geqslant 1$, note (as in the proof of Lemma 4.1) that K is a transverse push-off of L_m, so L_m must be non-loose. Then, since $\left|L_m \cap D\right| = 1$, $d(L_m) \leqslant 1$, and so we conclude that $d(L_m) = 1$. □

Finally, we use these destabilisations to define admissible transverse r-surgery on K, for $r \geqslant 0$.

Proof of Theorem 1.8. Ito and Kawamuro showed in [25, Corollary 3.4] that the binding of an open book (Σ, ϕ) is a transverse link of depth 1 if and only if the monodromy is not right-veering. With our hypotheses, we conclude that $d(K) = 1$.

Let (Σ, ϕ) be the open book defined by K. By Theorem 2.8, an open book for admissible transverse $(-1/n)$-surgery on K is given by $(\Sigma, \phi \circ \tau_\partial^n)$, where τ_∂ is a Dehn twist about the binding component of Σ. By Corollary 1.3, this supports a contact structure $\xi_{K_{-1/n}}$ that satisfies $c_{\mathrm{red}}^+(\xi_{K_{-1/n}}) \neq 0$, for sufficiently large integers n.

Since $d(K) = 1$, Lemma 4.2 guarantees the existence of Legendrian approximations L_m of K with $tb(L_m) = m$, for any m. We will use the neighbourhood of K coming from a standard neighbourhood of L_m to define transverse admissible r-surgery on K, for $r < m$. This neighbourhood also corresponds to a neighbourhood of $K_{-1/n} \subset (M_{-1/n}(K), \xi_{K_{-1/n}})$, and admissible transverse r-surgery on $K \subset (M, \xi_K)$ for $-1/n < r < m$ is equivalent to some admissible transverse surgery on $K_{-1/n} \subset (M_{-1/n}(K), \xi_{K_{-1/n}})$ using the corresponding neighbourhood. The theorem then follows from Lemma 2.9. $\square$

Proof of Corollary 1.10. The first statement, for $HF_{\mathrm{red}}(M) = 0$, follows from Theorems 1.1 and 1.8.

For the second statement in the theorem, consider now a general M. The manifold $M_0(K)$ supports a tight contact structure in both its orientations (see for example [21]). If $c_{\mathrm{red}}^+(\xi_K)$ and $c_{\mathrm{red}}^+(\xi_{\overline{K}})$ both vanish, then $c^+(\xi_{K_r})$ and $c^+(\xi_{\overline{K}_r})$ do not vanish for any $r < 0$, by Theorem 3.1. So assume $c_{\mathrm{red}}^+(\xi_K) \neq 0$. By Lemma 2.9, $c^+(\xi_{K_r}) \neq 0$ for all $r < 0$. By Corollary 3.2, at least one of $c^+(\xi_{K_r})$ and $c^+(\xi_{\overline{K}_{-r}})$ is non-vanishing for any $r > 0$, unless $r = R^+(K) = R_{\mathrm{red}}^+(K)$.

For the last statement, note that at least one of ϕ_K and $\phi_{\overline{K}}$ is not right-veering (unless $M = \#\left(S^1 \times S^2\right)$ and $\phi_K = \mathrm{id}$, whereupon $HF_{\mathrm{red}}(M) = 0$, and we're done, as above). Assume $\phi_{\overline{K}}$ is not right-veering, so $\xi_{\overline{K}}$ is overtwisted. Then $(M_r(K), \xi_{K_r})$ is tight for all $r < 0$, by Theorem 3.1, and $-(M_r(K)) = (-M)_{-r}(\overline{K})$ supports a tight contact structure for all $r < 0$, by Theorem 1.8. $\square$

If the open book (Σ, ϕ) is a negative stabilisation of another open book (whereupon it is not right-veering), Baker and Onaran explicitly identified an overtwisted disc that intersects K once and has boundary on the page of a stabilisation of (Σ, ϕ) (see [2, Theorem 5.2.3]). Given a transverse knot K, a Legendrian approximation L of K, and an overtwisted disc D such that $|L \cap D| = 1$, a careful tracking of dividing curves will show that contact (-1)-surgery on ∂D results in a contact manifold contactomorphic to the starting one, but where L gets identified with its negative destabilisation. We can use this and Baker and Onaran's identification of ∂D on a page to construct open books for admissible transverse r-surgery on K for $r \geqslant 0$.

Example 4.3. Let $K \subset (S^3, \xi_K)$ be the figure-eight knot. After a positive stabilisation using a boundary-parallel arc, we use [2, Theorem 5.2.3] to identify γ in Figure 3 as the boundary of an overtwisted disc D which K intersects once.

In the construction of S^3 from the open book decomposition $(\Sigma, \tau_\alpha^{-1} \tau_\beta \tau_{\delta_2})$, the boundary component parallel to δ_1 is identified with the figure-eight knot K, and the framing on K given by Σ is -1 (with respect to the Seifert framing). The open book $(\Sigma, \tau_\alpha^{-1} \tau_\beta \tau_\gamma^n \tau_{\delta_2})$, for some $n \in \mathbb{Z}$, also supports (S^3, ξ_K), but now the framing given to K by Σ is $n - 1$, since composing the monodromy with τ_γ^n is equivalent to contact (-1)-surgery on n push-offs of $\gamma = \partial D$. Thus, the open book $(\Sigma, \tau_\alpha^{-1} \tau_\beta \tau_\gamma^n \tau_{\delta_1} \tau_{\delta_2})$ supports the tight contact manifold $(S_{n-2}^3(K), \xi_{n-2})$, where ξ_{n-2} is the result of admissible transverse $(n-2)$-surgery on K.

5. FURTHER QUESTIONS

In this section, we discuss how Theorem 3.1 might be brought to bear on some open questions.
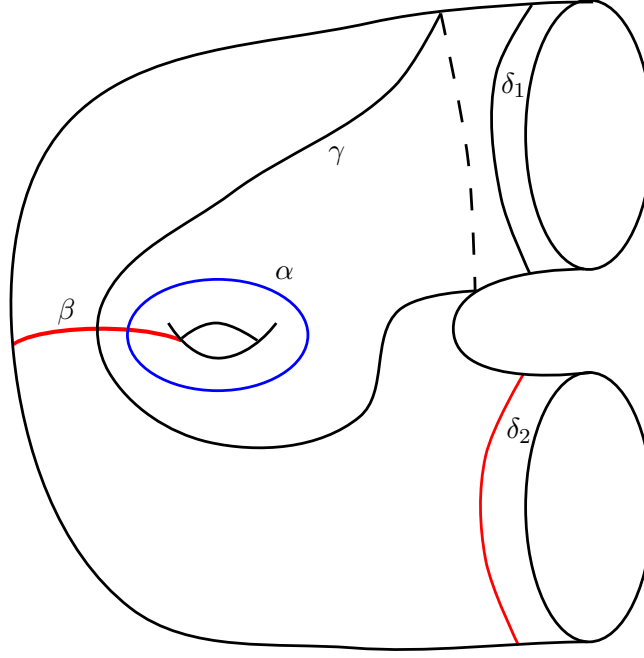
FIGURE 3. The open book $(\Sigma, \tau_\alpha^{-1} \tau_\beta \tau_{\delta_2})$ is a positive stabilisation of the open book given by the figure-eight knot $K \subset S^3$. The curve δ_1 is a push-off of K, and γ is the boundary of an overtwisted disc in ξ_K.

5.1. c^+ **versus** $\hat{c}$.
Given a contact manifold (M, ξ), we know that $c^+(\xi)$ vanishes if $\hat{c}(\xi)$ does. The converse, however, remains open: does the vanishing of $c^+(\xi)$ imply the vanishing of $\hat{c}(\xi)$?

To explore this, we define two invariants of elements of $\widehat{HF}(M)$, using $CF^{0,i}(M) = \ker U^i \subset CF^+(M)$, and its homology $HF^{0,i}(M)$.

Definition 5.1. Given $x \in \widehat{HF}(M)$, let the *vanishing order* $\mathrm{V}(x)$ of x be the minimum i such that the image of x in $\widehat{HF}(M) \to HF^{0,i}(M)$ is trivial (and it equals ∞ is no such finite i exists). Given $x \neq 0 \in \widehat{HF}(M)$, let the *dual vanishing order* $\widetilde{\mathrm{V}}(x)$ of x be the minimum of $\mathrm{V}(y)$ over all $y \in \widehat{HF}(-M)$ that pair non-trivially with x (and it equals ∞ if no such y exists).

Note that $\hat{c}(\xi) \neq 0$ if and only if $\mathrm{V}(\hat{c}(\xi)) > 0$; $c^+(\xi) \neq 0$ if and only if $\mathrm{V}(\hat{c}(\xi)) = \infty$; and $c_{\mathrm{red}}^+(\xi) \neq 0$ if and only if $\mathrm{V}(\hat{c}(\xi)) = \infty$ and $\widetilde{\mathrm{V}}(\hat{c}(\xi)) < \infty$.

Let $K \subset M$ be a fibred knot, fix an integer $m \geqslant 1$, and consider the following short exact sequences, where the bottom-right complex $C = \mathcal{C}_{\overline{K}}\{1 \leqslant \max(i, j - g + 1) \leqslant m\} \cup \mathcal{C}_{\overline{K}}\{i = m, j = g + m\}$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}_{\overline{K}}\{i = 0, j \leqslant g - 1\} & \longrightarrow & \mathcal{C}_{\overline{K}}\{i = 0\} & \longrightarrow & \mathcal{C}_{\overline{K}}\{i = 0, j = g\} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}_{\overline{K}}\{i = 0, j \leqslant g - 1\} & \longrightarrow & \mathcal{C}_{\overline{K}}\{0 \leqslant i \leqslant m\} & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$

Then, the same proof as in Theorem 3.1(1), and the similar version corresponding to Theorem 3.1(2), gives the following.

Theorem 5.2. *Let $K \subset M$ be a fibred knot. Then for all sufficiently negative integers n:*

(1) $\widetilde{V}(\widehat{c}(\xi_{\overline{K}})) - 1 \leqslant V(\widehat{c}(\xi_{K_n})) \leqslant \widetilde{V}(\widehat{c}(\xi_{\overline{K}}))$.
(2) $\widetilde{V}(\widehat{c}(\xi_{K_n})) - 1 \leqslant V(\widehat{c}(\xi_{\overline{K}})) \leqslant \widetilde{V}(\widehat{c}(\xi_{K_n}))$.

Remark 5.3. The inequalities are there because C is not precisely $\ker(U|_{A^+})^i$ for any i. In fact, with the identification of A^+ with $\mathcal{C}_{\overline{K}}\{\max(i, j-g+1) \geqslant 1\}$, we have $\ker(U|_{A^+})^m \subsetneq C \subsetneq \ker(U|_{A^+})^{m+1}$.

If we could find a fibred knot K supporting (M, ξ_K), such that $c_{\mathrm{red}}^+(\xi_K) \neq 0$ and $\widetilde{V}(\xi_K) > 1$, then Theorem 5.2 would imply that for all sufficiently negative integers n, $c^+(\xi_{K_n}) = 0$, but $\widehat{c}(\xi_{\overline{K}_n}) \neq 0$.

Question 5.4. *Do there exist examples of contact manifolds (M, ξ) with $\widetilde{V}(\widehat{c}(\xi)) > 1$? Is there any geometric meaning to $\widetilde{V}(\widehat{c}(\xi))$?*

5.2. **Genus-1 Open Books.**

As yet no obstruction has been found for a non-planar contact manifold (M, ξ) to be supported by a genus-1 open book decomposition. In fact, it is currently unknown whether there exist any contact manifolds whose *support genus* (*ie.* minimal genus of a supporting open book decomposition, defined in [13]) is greater than 1. To explore this, we look at properties of genus-1 open books with one boundary component.

Let $K \subset M$ be a fibred knot of genus 1. A result of Honda, Kazez, and Matić ([24]), and independently Baldwin ([3]), shows that ξ_K is tight if and only if $c^+(\xi_K)$ is non-vanishing. Honda, Kazez, and Matić prove that this is equivalent to the monodromy ϕ_K of the open book having *fractional Dehn twist coefficient* $\mathrm{FDTC}(\phi_K)$ greater than 0, or greater-than-or-equal to 0 if ϕ_K is a periodic mapping class (see [24] for details on these terms).

Theorem 5.5. *Let $K \subset M$ be a fibred knot of genus 1, and let $(M_r(K), \xi_{K_r})$ be the result of admissible transverse r-surgery on $K \subset (M, \xi_K)$, for $r < 0$. The following are equivalent:*

(1) $c_{\mathrm{red}}^+(\xi_{\overline{K}}) \neq 0$ and $\widetilde{V}(\widehat{c}(\xi_{\overline{K}})) = 1$.
(2) $c^+(\xi_{K_r}) = 0$ *for some $r < 0$.*
(3) $\widehat{c}(\xi_{K_r}) = 0$ *for some $r < 0$.*
(4) $c^+(\xi_{K_{-1}}) = 0$.
(5) $\widehat{c}(\xi_{K_{-1}}) = 0$.
(6) $\xi_{K_{-1}}$ *is overtwisted.*
(7) (a) $\mathrm{FDTC}(\phi_{\overline{K}}) > 1$, *if $\phi_{\overline{K}}$ is periodic.*
     (b) $\mathrm{FDTC}(\phi_{\overline{K}}) \geqslant 1$, *if $\phi_{\overline{K}}$ is not periodic.*

Proof. (1) $\implies$ (2): This is Theorem 3.1.

(4) $\iff$ (5) $\iff$ (6): These follow from the discussion preceding the theorem.

(5) $\implies$ (3) $\implies$ (2): These are trivial.

(2) $\implies$ (4): We show the contrapositive. We claim that inadmissible transverse $\frac{n}{n-1}$-surgery on K_{-1}, for $n > 1$ an integer, results in the same contact manifold as admissible transverse $(-n)$-surgery on K, namely $(M_{-n}(K), \xi_{K_{-n}})$ (*cf.* [7, Section 3.3]). Indeed, let λ be the 0-framing on a neighbourhood of K, and let μ be a meridional curve for K. Then the page of the open book for K_{-1} also traces the curve λ on the boundary of a neighbourhood of K_{-1}, but now the meridional curve is $\mu - \lambda$. Inadmissible transverse $\frac{n}{n-1}$-surgery on K_{-1}, measured in terms of λ and μ, is

$$n \cdot (\mu - \lambda) + (n-1) \cdot \lambda = n \cdot \mu - \lambda,$$

which is $(-n)$-surgery on K. That the result of inadmissible transverse surgery K_{-1} can be considered as a single transverse surgery on K follows from the definition of transverse surgery.

By [7, Theorem 1.6], if $c^+(\xi_{K_{-1}}) \neq 0$, then inadmissible transverse r-surgery on K_{-1} preserves this non-vanishing for all $r > 1$. In particular, $c^+(\xi_{K_{-n}}) \neq 0$ for all integers $n > 0$. By Lemma 2.9, this implies that $c^+(\xi_{K_r}) \neq 0$ for all $r < 0$.

(3) $\implies$ (1): The only part of this that does not follow from Theorem 3.1 is the dual vanishing order of $\widehat{c}(\xi_{\overline{K}})$. However, if $\widehat{c}(\xi_{K_r}) = 0$ some $r < 0$, then Lemma 2.9 implies that $\widehat{c}(\xi_{K_n}) = 0$ for all

sufficiently negative integers n, *ie.* $\mathrm{V}(\widehat{c}(\xi_{K_n})) = 0$. Thus, by Theorem 5.2, $\widetilde{\mathrm{V}}(\widehat{c}(\xi_{\overline{K}})) \leqslant 1$. However, since $c^+_{\mathrm{red}}(\xi_{\overline{K}}) \neq 0$, we know that $\widetilde{\mathrm{V}}(\widehat{c}(\xi_{\overline{K}})) \geqslant 1$, and so therefore it is equal to 1.

(4) $\iff$ (7): Note that an open book decomposition for $(M_{-1}(K), \xi_{K_{-1}})$ is given by $(\Sigma_{1,1}, \phi_K \circ \tau_\partial)$, where $\Sigma_{1,1}$ is genus 1 surface with one boundary component, and τ_∂ is a positive Dehn twist about a curve parallel to $\partial\Sigma_{1,1}$. Furthermore, $\mathrm{FDTC}(\phi_K \circ \tau_\partial) = \mathrm{FDTC}(\phi_K) + 1$ and $\mathrm{FDTC}(\phi_K) = -\mathrm{FDTC}(\phi_{\overline{K}})$, according to [26]. Thus, the equivalence follows from the discussion preceding this theorem. $\square$

Consider now a contact manifold (M, ξ), supported by a genus-1 open book (Σ, ϕ), where $c^+_{\mathrm{red}}(\xi) \neq 0$. We can cap off (Σ, ϕ) to an open book $(\Sigma_{1,1}, \phi')$ with a single binding component that supports (M', ξ'), where $c^+_{\mathrm{red}}(\xi') \neq 0$, $\widetilde{\mathrm{V}}(\widehat{c}(\xi')) = 1$, and $\mathrm{FDTC}(\phi') > 1$, by Theorem 5.5.

Question 5.6. *By [5, Theorem 1.2], we know that $\widetilde{\mathrm{V}}(\widehat{c}(\xi)) \geqslant \widetilde{\mathrm{V}}(\widehat{c}(\xi'))$. Under what conditions can we say that $\widetilde{\mathrm{V}}(\widehat{c}(\xi')) = 1$ implies $\widetilde{\mathrm{V}}(\widehat{c}(\xi)) = 1$? Under what conditions does $\mathrm{FDTC}(\phi') > 1$ imply that $\mathrm{FDTC}(\phi, \mathrm{B}) > 1$ at every boundary component B of Σ?*

Even partial answers to Question 5.6 could lead to obstructions to ξ being supported by a genus-1 open book.

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