Math 53 Spring 2016
Quiz 4 2/29/16 Name: **Solutions**

You have 15 minutes to take this quiz. Do not use notes, calculators, or classmates to assist you.

1. Consider the function $f(x, y, z) = x^3 + y^3 - z^2 + (x - 1)z$.

a) Compute the gradient of f.

b) Find an equation of the tangent plane to the level surface $f(x, y, z) = 1$ at the point $(1, 1, 1)$.

a) This is a straightforward computation:

$$\nabla f(x, y, z) = \langle 3x^2 + z, 3y^2, -2x + x - 1\rangle$$

b) Recall that the gradient is always normal to the level surfaces, which in particular means that it is normal to the tangent plane to the level surface. The tangent plane is thus the plane through $(1, 1, 1)$ normal to $\nabla f(1, 1, 1) = \langle 4, 3, -2\rangle$, so it has equation

$$4(x - 1) + 3(y - 1) - 2(z - 1) = 0$$

2. Consider the related function $g(x, y) = x^3 - y^2 + (x - 1)y$. Find the critical points of g and use the second derivative test to classify them as minima, maxima, or saddle points. (If the second derivative test is not sufficient to classify a point, just say so: no need to try other techniques.)

Solution The critical points of g are the points where ∇g vanishes, so we first compute

$$\nabla g(x, y) = \langle 3x^2 + y, -2y + x - 1\rangle$$

Solving $\nabla g(x, y) = 0$ means solving $3x^2 + y = 0, -2y + x - 1 = 0$. One way to do this is to rewrite these equations as $6x^2 + 2y = 0$ and $2y = x - 1$, which together give $6x^2 + x - 1$, which has solutions

$$\frac{-1 \pm \sqrt{1 + 4 \cdot 6}}{12} = \frac{-1 \pm 5}{12} = -\frac{1}{2}, \frac{1}{3}$$

Using $y = (x - 1)/2$, the critical points are $(-1/2, -3/4)$ and $(1/3, -1/3)$. To classify them, we look at the Hessian matrix

$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix} = \begin{bmatrix} 6x & 1 \\ 1 & -2 \end{bmatrix}$$

Its determinant is $-12x - 1$. At $(-1/2, -3/4)$, the determinant is positive and g_{xx} is negative, and we conclude that $(-1/2, -3/4)$ is a minimum. At $(1/3, -1/3)$ the determinant is negative, so $(1/3, -1/3)$ is a saddle point.