The Langlands-Kottwitz method and trace formulae

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1 How do we get information out of Matsushima?

This talk is based on §2 of Youcis' notes.

Let G be a "sufficiently nice" (yields a Shimura variety, G^{der} is $\mathbb{Q}$-anisotropic) reductive group. Recall Matsushima's formula:

$H^i_{\mathrm{et}}(\mathrm{Sh}_G, \mathscr{F}_\xi \otimes \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\pi^\infty} \pi^\infty \otimes \sigma^i(\pi^\sigma)$ where π^∞ ranges over the finite parts of automorphic representations (with a suitable central character), $\mathscr{F}_\xi$ is a local system on Sh_G assocated to some algebraic representation $\xi : G \to \mathrm{GL}(V)$, and $\sigma^i(\pi^\sigma)$ is some multiplicity factor on which the Galois group Γ_E acts, where E is the reflex field of the Shimura variety (over which it is defined).

We would really like ways to analyze the Galois representations $\sigma^i(\pi^\infty)$: what are its properties (e.g. trace, determinant) in terms of π^∞, and is it related to the Langlands conjecture?

The trace, in particular, is of interest due to:

Theorem 1.0.1. (Brauer-Nesbitt.) *A Galois representation $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, unramified outside a finite set of primes, is uniquely determined by the values of the traces of $\rho(\mathrm{Frob}_p)$ for unramified primes p.*

We let $H^*(\mathrm{Sh}_G, \mathscr{F}_\xi := \sum_{i=0}^{2\dim X_G}(-1)^i H^i(\mathrm{Sh}_G, \mathscr{F}_\xi)$ in the Grothendieck group of $G(\mathbb{A}_f) \times \Gamma_E$. For every π^∞, This yields a *virtual representation*

$$\sigma^*(\pi^\infty) = \sum_{i=0}^{2\dim X_G} (-1)^i \sigma^i(\pi^\infty)$$

of the Galois group. This is a formal difference of Galois representations, and the trace of a virtual representation is the different of the traces of its positive and negative components. Ideally, all but one of the terms in this sum will vanish (in particular, the $i = \dim \mathrm{Sh}(G)$ term will be the only nonzero term).

Recall that $G(\mathbb{A}_f)$-representations are essentially equivalent to representations of the Hecke algebra $\mathscr{H}(G(\mathbb{A}_f))$, so we study the trace of an element of the form $\tau \times f \in \Gamma_E \times \mathscr{H}(G(\mathbb{A}_f))$. For a given automorphic representation π, it turns out that we can choose

$f = f_0$ to be a projector onto the π_0^∞ component of $H^i(\mathrm{Sh}_G, \mathscr{F}_\xi)$. Thus, the trace of $\tau \times f_0$ on $H^*(\mathrm{Sh}_G, \mathscr{F}_\xi)$ is $\mathrm{tr}(\tau|\sigma^*(\pi_0^\infty))$, the trace of the Galois action on the virtual representation $\sigma^*(\pi_0^\infty)$. Therefore it behooves us to give "useful" trace formulae for the action of $\tau \times f$.

1.1 Arthur-Selberg trace formula

"Useful" usually means "expressed in terms of an orbital integrals." The reason that this is useful is the Arthur-Selberg trace formula:

Theorem 1.1.1. *(Arthur-Selberg trace formula, simply connected and anisotropic case.) Let $f \in \mathscr{H}_{\mathbb{C}}(G(\mathbb{A}_f), \chi^{-1})$. Then*

$$\sum_{\pi} m(\pi)\,\mathrm{tr}(f|_\pi) = \sum_{\{\gamma\}} v_\gamma O_\gamma(f)$$

where π runs over all automorphic representations for G which central character χ, $\{\gamma\}$ runs through all conjugacy class in $G(\mathbb{Q})$, v_γ is some volume term, and

$$O_\gamma := \int_{\mathrm{Stab}_{G(\mathbb{A}(f))}(\gamma)\backslash G(\mathbb{A}_f)} f(g^{-1}\gamma g)\, dg.$$

So the goal becomes to express the trace of $\tau \times f$ in terms of (sums of) orbital integrals—that way, we can use the trace formula to relate this back to traces of f.

1.2 The Langlands-Kottwitz method

This is very doable for PEL Shimura varieties, which are roughly moduli spaces of abelian varieties with extra structure. However, we need some restrictions on the element $\tau \times f^\infty$: first, τ needs to lie in some Weil group $W_{E_\mathfrak{p}}$ for some prime $\mathfrak{p}$ of E, and f needs to be of the form $f^p 1_{\mathcal{G}(\mathbb{Z}_p)}$ where f^p is defined away from p and $\mathcal{G}(\mathbb{Z}_p)$ is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. These restrictions mean that we can think of the $\tau \times f^p$ action as an action on $H^*(\mathrm{Sh}_G(K^p\mathcal{G}(\mathbb{Z}_p))_{E_\mathfrak{p}}, \mathscr{F}_\xi)$, where K^p is some compact open subgroup of $G(\mathbb{A}_f^p)$ underwhich f^p is bi-invariant. The reason that we want to look at this variety $\mathrm{Sh}_G(K^p\mathcal{G}(\mathbb{Z}_p))_{E_\mathfrak{p}}$ is because it has good reduction at p (due to the hyperspecial subgroup); that is, there is a smooth proper canonical model $\mathscr{S}_G(K^p)$ of this variety over $E_\mathfrak{p}$. We can then use standard proper base change formulas to related the trace on the old Shimura variety to the reduction, i.e. on $H^*(\mathscr{S}_G(K^p)_{\overline{\mathbb{F}}_p}, \mathscr{F}_\xi)$. Such a trace can be realized as a trace of a correspondence and is computed via a generalization of the Grothendieck-Lefschetz trace formula, which boils down to counting fixed points under the correspondence.

Example 1.2.1. Even though this doesn't meet our assumptions, consider $G = \mathrm{GL}_2$. Then we are trying to count elliptic curves over finite fields with level structure. Honda-Tate theory tells us that isogeny classes of elliptic curves are in bijection with Weil q-polynomials, which are in bijection with certain semisimply conjugacy classes in $\mathrm{GL}_2(\mathbb{Q})$. Counting elliptic curves up to isomorphism within an isogeny class, say of a fixed curve E_0, means counting isomorphism classes of lattices in $\hat{V}(E_0)$, which are given by some orbital integral. So the sum in the trace formula should be thought of as counting isogeny classes, while the weighted orbital integrals count isomorphism classes within isogeny classes.

Alan will discuss this example in more detail in his talk.