Reference sheet for classical roots systems

CJ Dowd

September 21, 2023

This reference sheet was made in preparation for my qualifying exam. All errors are my own; let me know if you find any. Please feel free to copy/distribute if this reference sheet is helpful to you.

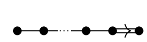
Notation: e_{ij} denotes the matrix unit, where the only nonzero entry is a 1 in row i, column j. We use the shorthand $e_i := e_{ii}$ for the diagonal unit matrices. ϵ_i is the linear functional defined on the diagonal by $\epsilon_i(e_j) = \delta_{ij}$. X^R denotes the transpose of a matrix along its antidiagonal.

There are always many choices involved in determining the data listed below. For example, there are other common presentations of $\mathfrak{so}_N$ and $\mathfrak{sp}_{2n}$ depending on the specific choice of symmetric/antisymmetric invariant form, and there are many choices of simple roots, differing by the action of the Weyl group.

1 $\mathfrak{sl}_n(\mathbb{C})$

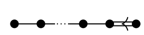
- Isomorphic to complexification of $\mathfrak{su}(n)$
- $\mathbb{C}$-dimension: $n^2 - 1$
- Subalgebra of $\mathfrak{gl}_n(\mathbb{C})$ consisting of traceless matrices
- CSA: traceless diagonal matrices $\sum_{i=1}^n a_i e_i$ with $\sum a_i = 0$
- Rank: $n - 1$
- Roots: $\alpha_{ij} = \epsilon_i - \epsilon_j$ $(i \neq j)$
- Simple roots: $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(1 \leq i \leq n-1)$. All roots have the same length.
- Simple coroots: $e_i - e_{i+1,i+1}$ $(1 \leq i \leq n-1)$
- Root spaces: $\mathfrak{g}_{\alpha_{ij}} = \langle e_{ij} \rangle$ $(i \neq j)$
- Fundamental dominant weights: $\epsilon_1, \epsilon_1 + \epsilon_2, \ldots, \sum_{i=1}^{n-1} \epsilon_i$
- Cartan matrix: $$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$
- Dynkin diagram A_{n-1}: . Mnemonic: "$\mathfrak{sl}_n$ is the first semisimple Lie algebra you ever learn about, so its Dynkin diagram comes first in the alphabet."
- Weyl group: S_n. Acts by permuting the ϵ_i.
- Highest weight of standard representation: ϵ_1
- Highest weight of adjoint representation: $\epsilon_1 - \epsilon_n = \epsilon_1 + \sum_{i=1}^{n-1} \epsilon_i$

2 $\mathfrak{so}_{2n+1}(\mathbb{C}), n \geq 2$

- Isomorphic to complexification of $\mathfrak{so}(m, 2n+1-m)$ for any $0 \leq m \leq 2n+1$.
- $\mathfrak{so}_5(\mathbb{C}) \simeq \mathfrak{sp}_4(\mathbb{C})$
- $\mathbb{C}$-dimension: $n(2n+1)$
- Subalgebra of $\mathfrak{gl}_{2n+1}(\mathbb{C})$ of matrices that are antisymmetric along the antidiagonal: $X = -X^R$
- Cartan subalgebra: diagonal matrices of the form $\begin{pmatrix} D & & \\ & 0 & \\ & & -D^R \end{pmatrix}$, where D is an $n \times n$ diagonal matrix.
- Rank: n
- Roots: $\pm(\epsilon_i \pm \epsilon_j)$, $1 \leq i \neq j \leq n$, as well as $\pm\epsilon_i$ for $1 \leq i \leq n$
- Simple roots: $\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n$. Last root is the unique short simple root.
- Simple coroots: $e_i - e_{2n-i+2} - e_{i+1,i+1} + e_{2n-i+1,2n-i+1}$, and also $2e_n - 2e_{n+2}$.
- Root spaces: for $i + j \leq n$, the root space $\mathfrak{g}_{\epsilon_i - \epsilon_j}$ is spanned by $e_{ij} - e_{2n-i+2,2n-j+2}$, and the root space $\mathfrak{g}_{\epsilon_i + \epsilon_j}$ is spanned by $e_{i+n,j} - e_{n-i+2,2n-j+2}$. For $1 \leq i \leq n$, root space $\mathfrak{g}_{\epsilon_i}$ is spanned by $e_{i,n+1} - e_{2n-i+2,n+1}$, and root space $\mathfrak{g}_{-\epsilon_i}$ spanned by $e_{n+1,i} - e_{n+1,2n-i+2}$.
- Fundamental dominant weights: $\epsilon_1, \epsilon_1 + \epsilon_2, \ldots, \sum_{i=1}^{n-1} \epsilon_i$, $\frac{1}{2}\sum_{i=1}^{n} \epsilon_i$
- Cartan matrix: $\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}$
- Dynkin diagram B_n: . Mnemonic: "B is for bizarre, which means *odd*, for the odd orthgonal Lie algebras." For the direction of the arrow: "All but one of the simple roots are **B**ig."
- Weyl group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$, where (τ, σ) acts by swapping the signs of the ϵ_i corresponding to the nonzero entries of τ and then permuting the results under σ.
- Highest weight of standard representation: ϵ_1
- Highest weight of adjoint representation: $\epsilon_1 + \epsilon_2$

3 $\mathfrak{sp}_{2n}(\mathbb{C}), n \geq 2$

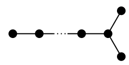
- Isomorphic to complexification of $\mathrm{Sp}(n)$
- $\mathfrak{sp}_4(\mathbb{C}) \simeq \mathfrak{so}_5(\mathbb{C})$
- $\mathbb{C}$-dimension: $2n^2 + n$
- Subalgebra of $\mathfrak{gl}_{2n}(\mathbb{C})$ consisting of matrices of the form $\begin{pmatrix} A & B \\ C & -A^R \end{pmatrix}$, $B = B^R, C = C^R$, all blocks are $n \times n$.

- Cartan subalgebra: diagonal matrices of the form $\begin{pmatrix} D & \\ & -D^R \end{pmatrix}$
- Rank: n
- Roots: $\pm(\epsilon_i \pm \epsilon_j)$ for $1 \leq i \neq j \leq n$ (same as $\mathfrak{so}_{2n}$). Also $\pm 2\epsilon_i$ for $1 \leq i \leq n$
- Simple roots: $\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n$, and also $2\epsilon_n$. Last root is the only long simple root.
- Simple coroots: for $e_i - e_{i+1} - e_{2n-i+1} + e_{2n-i}$ for $1 \leq i \leq n-1$, and $e_n - e_{n+1}$
- Fundamental weights: $\epsilon_1, \epsilon_1 + \epsilon_2, \ldots, \sum_{i=1}^{n} \epsilon_i$.
- Cartan matrix: $$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix}$$
- Dynkin diagram C_n: . Mnemonic: "C is for **C**ymplectic." For the direction of the arrow: "All but one of the simple roots are small, like a small **C**at."
- Weyl group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$, where (τ, σ) acts by swapping the signs of the ϵ_i corresponding to the nonzero entries of τ and then permuting the results under σ. (Similar to $\mathfrak{so}_{2n+1}$.)
- Highest weight of standard representation: ϵ_1
- Highest weight of adjoint representation: $\epsilon_1 + \epsilon_2$

4 $\mathfrak{so}_{2n}(\mathbb{C})$, $n \geq 3$

- Isomorphic to $\mathfrak{so}(m, 2n-m)_{\mathbb{C}}$ for any $0 \leq m \leq 2n$
- $\mathfrak{so}_4(\mathbb{C}) \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{so}_6(\mathbb{C}) \simeq \mathfrak{sl}_3(\mathbb{C})$
- $\mathbb{C}$-dimension: $2n^2 - n$
- Subalgebra of $\mathfrak{gl}_{2n}(\mathbb{C})$ of matrices that are antisymmetric along the antidiagonal: $X = -X^R$
- Cartan subalgebra: diagonal matrices of the form $\begin{pmatrix} D & \\ & -D^R \end{pmatrix}$
- Rank: n
- Roots: $\pm(\epsilon_i \pm \epsilon_j)$ $(i \neq j)$
- Simple roots: $\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n$. All roots have the same length.
- Simple coroots: $e_i - e_{i+1,i+1} - e_{n-i+1} + e_{n-i}$ for $1 \leq i \leq n-1$, and also $e_{n-1} + e_n - e_{n+1} - e_{n+2}$
- Root spaces: for $1 \leq i \neq j \leq n$, the root space $\mathfrak{g}_{\epsilon_i - \epsilon_j}$ is spanned by $e_{ij} - e_{2n-i+1,2n-j+1}$, and the root space $\mathfrak{g}_{\epsilon_i + \epsilon_j}$ is spanned by $e_{n+i,j} - e_{n-i+1,2n-j+1}$
- Fundamental dominant weights: $\epsilon_1, \epsilon_1 + \epsilon_2, \ldots, \sum_{i=1}^{n-1} \epsilon_i$, $\frac{1}{2} \sum_{i=1}^{n} \epsilon_i$

- Cartan matrix: $\begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & \ddots & & & \\ & & \ddots & \ddots & -1 & & \\ & & & -1 & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix}$

- Dynkin diagram D_n: . Mnemonic: "D is for **D**ouble, for the even orthogonal Lie algebras."

- Weyl group $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes A_n$, where we treat $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ as the trace 0 hyperplane inside $(\mathbb{Z}/2\mathbb{Z})^n$, and then apply the same action as in the $\mathfrak{so}_{2n+1}$ case.

- Highest weight of standard representation: ϵ_1

- Highest weight of adjoint representation: $\epsilon_1 + \epsilon_2$