A note on the projection formula

CJ Dowd

August 9, 2023

Let $f : X \to Y$ be a morphism of schemes, let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and let $\mathscr{E}$ be locally free sheaf of finite rank on Y. The projection formula gives a natural isomorphism

$$R^i f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E}) \simeq (R^i f_*\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{E},$$

where $R^i f_*$ is the higher pushforward. For $i = 0$, this means

$$f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E}) \simeq (f_*\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{E}.$$

The formula is rather abstract-looking. For a long time, I had a hard time gaining the intuition behind it, and I did not understand its importance. Then, while working on Hartshorne Exercise III.5.6, Feiyang Lin (my qualifying exam study partner) pointed out that our work could only be made rigorous by applying the projection formula. I realized that I had been frequently using the formula implicitly in a very common setting.

Let $f : X \hookrightarrow Y$ be a closed embedding of projective varieties over a field, and let $\mathscr{O}_Y(1)$ be a very ample sheaf on Y. This sheaf pulls back to a very ample sheaf $\mathscr{O}_X(1) := f^*\mathscr{O}_Y(1)$. The closed subscheme exact sequence is

$$0 \to \mathscr{I}_X \to \mathscr{O}_Y \to f_*\mathscr{O}_X \to 0,$$

where $\mathscr{I}_X$ is the ideal sheaf associated to X. We usually abuse notation by writing $\mathscr{O}_X$ instead of $f_*\mathscr{O}_X$ in this exact sequence, which might add to some of the confusion.

Very often we like to *twist* this exact sequence by $\mathscr{O}_Y(n) = \mathscr{O}_Y(1)^{\otimes n}$. We write this as

$$0 \to \mathscr{I}_X(n) \to \mathscr{O}_Y(n) \to (f_*\mathscr{O}_X)(n) \to 0$$

But we would like to understand this last term as the pushforward of a sheaf on X, rather than a pushforward followed by a tensor product. The projection formula lets us do precisely this: we have

$$(f_*\mathscr{O}_X) \otimes \mathscr{O}_Y(n) \simeq f_*(\mathscr{O}_X \otimes f^*\mathscr{O}_Y(n)) = f_*(\mathscr{O}_X(n)),$$

where by definition $\mathscr{O}_X(n) = f^*\mathscr{O}_Y(n)$. Since cohomology commutes with pushforward by a closed embedding, we can understand a lot about this exact sequence if we understand $\mathscr{O}_X(n)$.

For example, let X be a smooth non-hyperelliptic curve of genus 4 over k, and let $f : X \hookrightarrow \mathbb{P}^3_k$ be its canonical embedding, so that $\mathscr{O}_X(1) = \omega_X$. Then using the projection formula, twisting the closed subscheme exact sequence yields

$$0 \to \mathscr{I}_X(n) \to \mathscr{O}_{\mathbb{P}^3_k}(n) \to f_*\omega_X^n \to 0$$

for any integer n. But we know a lot about the cohomology groups $H^i(X, \omega_X^n) \simeq H^i(\mathbb{P}^3_k, f_*\omega_X^n)$ from the Riemann-Roch formula. For example, if $n = 2$, then $\dim_k H^0(X, \omega_X^2) = 4g - 4 + 1 - g = 9$. Taking global sections of the exact sequence, we can use this to conclude that $\mathscr{I}_X(2)$ contains a nonzero quadric Q, which must be unique up to scalars and irreducible since $f(X)$ is not contained in any hyperplane and X cannot be contained in the complete intersection of two quadrics, as X has degree 6 under the embedding f whereas such an intersection has degree 4. Likewise, $\dim_k H^0(X, \omega_X^3) = 15$ lets us deduce that $\mathscr{I}_X(3)$ contains five independent cubics, hence at least one cubic P not contained in the quadric Q. It then follows that X is the complete intersection of P and Q: X is certainly contained in this intersection, and this intersection has no embedded components due to the unmixedness theorem, so it must equal X.

Another important example is the smooth quadric surface $Q = V(xw - yx) \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$, via the Segre embedding $f : Q \hookrightarrow \mathbb{P}^3_k$. We have $\operatorname{Pic} Q \simeq \mathbb{Z}^2$, so we represent divisors as pairs of integers (a, b); the class $(a, 0)$ is represented by any a copies of the first $\mathbb{P}^1_k$ inside Q. We can compute $f^* \mathscr{O}_{\mathbb{P}^3_k}(n) = \mathscr{O}_Q(n, n)$. Let $n \geq 0$ be an integer. We have a closed subscheme exact sequence

$$0 \to \mathscr{O}_Q(-n, 0) \to \mathscr{O}_Q \to \iota_* \mathscr{O}_{(\mathbb{P}^1)^{\amalg} n} \to 0.$$

where $(\mathbb{P}^1)^{\amalg} n$ is n disjoint copies of $\mathbb{P}^1$ lying inside $\mathbb{P}^1$ (assume k is an infinite field). The pullback of (a, b) to a copy of $\mathbb{P}^1$ inside Q of the form $\mathbb{P}^1 \times \{*\}$ is $\mathscr{O}_{\mathbb{P}^1}(b)$, so the projection formula tells us that the above exact sequence twists to

$$0 \to \mathscr{O}_Q(-n + a, b) \to \mathscr{O}_Q(a, b) \to \mathscr{O}_{\mathbb{P}^1_k}(b)^{\oplus n} \to 0.$$

Now this tells us a lot about the cohomology of $\mathscr{O}_Q(a, b)$, namely the results of Hartshorne Exercise III.5.6. To start, note that we have an exact sequence

$$0 \to H^0(Q, \mathscr{O}_Q(-n, 0)) \to H^0(Q, \mathscr{O}_Q) \to H^0(\mathscr{O}_{\mathbb{P}^1_k})^{\oplus n} \to H^1(Q, \mathscr{O}_Q(-n, 0)) \to H^1(Q, \mathscr{O}_Q) = 0,$$

noting that $H^1(Q, \mathscr{O}_Q) = 0$ because it is a hypersurface in $\mathbb{P}^3_k$ (Hartshorne Exercise III.5.5). For $n \geq 1$, the map $H^0(Q, \mathscr{O}_Q) \to H^0(\mathscr{O}_{\mathbb{P}^1_k})^{\oplus n}$ may be identified with the diagonal map $k \hookrightarrow k^n$, so we conclude that $\dim H^0(Q, \mathscr{O}_Q(-n, 0)) = 0$ and $\dim H^1(Q, \mathscr{O}_Q(-n, 0)) = n - 1$ whenever $n \geq 1$. Now letting n be an arbitrary integer, we have another exact sequence

$$\begin{aligned} 0 \to H^0(Q, \mathscr{O}_Q(n-1, n)) \to H^0(Q, \mathscr{O}_Q(n, n)) \to H^0(\mathscr{O}_{\mathbb{P}^1_k}(n)) \to H^1(Q, \mathscr{O}_Q(n-1, n)) \\ \to H^1(Q, \mathscr{O}_Q(n, n)) = 0 \end{aligned}$$

where again $H^1(Q, \mathscr{O}_Q(-n, -n)) = 0$ by III.5.5(c), since $\mathscr{O}_Q(n, n) = f^* \mathscr{O}_{\mathbb{P}^3_k}(n)$. The map $H^0(Q, \mathscr{O}_Q(n, n)) \to H^0(\mathscr{O}_{\mathbb{P}^1_k}(n))$ is always surjective—the former is spanned by monomials in $\{x_0, x_1, y_0, y_1\}$ of bidegree (n, n), and the latter consists of monomials in $\{x_0, x_1\}$ of degree n, and pullback of global sections is given by setting y_0 and y_1 to (not both simultaneously zero) constants. We conclude that $H^1(Q, \mathscr{O}_Q(a, b)) = 0$ whenever $|a - b| \leq 1$. Finally, for $a, b > 0$, we have an exact sequence

$$0 \to H^0(Q, \mathscr{O}_Q(-a, -b)) \to H^0(Q, \mathscr{O}_Q) \to H^0(\mathscr{O}_Y) \to H^1(Q, \mathscr{O}_Q(-a, -b)) \to H^1(Q, \mathscr{O}_Q) = 0,$$

where Y is the subscheme of Q given by the union of a disjoint copies of the first $\mathbb{P}^1$ and b disjoint copies of the second $\mathbb{P}^1$; this is a connected reduced projective scheme when both $a, b > 0$, so its global sections are just k, and we conclude that $H^1(Q, \mathscr{O}_Q(-a, -b)) = 0$.

These cohomological facts tell us a lot about curves of type (a, b) on Q. For example, vanishing of $H^1(Q, \mathscr{O}_Q(a, b))$ for $a, b > 0$ implies that *any* curve of type (a, b) is connected via the long exact sequence in cohomology; note that we deduced this fact from the case of a specific curve of type (a, b), and then the cohomological machinery implies that it works for any representative.

Another fact is that a regular curve C of type (a, b) is projectively normal (as a subscheme of $\mathbb{P}^3_k$) if and only if $|a - b| \leq 1$. By Hartshorne Exercise II.5.14(d), this is equivalent to $H^0(\mathbb{P}^3_k, \mathscr{O}(n)) \to H^0(C, \mathscr{O}_C(n))$ being surjective. The map $H^0(\mathbb{P}^3_k, \mathscr{O}(n)) \to H^0(Q, \mathscr{O}(n, n))$ is always surjective since $H^1(\mathbb{P}^3_k, \mathscr{O}(-2 + n)) = 0$—note that $\mathscr{I}_Q \simeq \mathscr{O}_{\mathbb{P}^3_k}(-2)$. The map $H^0(Q, \mathscr{O}(n, n)) \to H^0(C, \mathscr{O}_C(n))$ is surjective if $|a - b| \leq 1$ by vanishing of $H^1(Q, \mathscr{O}_Q(n - a, n - b))$. Otherwise, if $a > b + 1$, nonvanishing of $H^1(-a + b, 0)$ implies that $H^0(Q, \mathscr{O}(b, b)) \to H^0(C, \mathscr{O}_C(b))$—hence also $H^0(\mathbb{P}^3_k, \mathscr{O}(b)) \to H^0(C, \mathscr{O}_C(b))$—is not surjective.