Computations on parabolic cohomology for modular curves

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The reference for today's talk is Hida's *Elementary Theory of Eisenstein Series and L-functions*, §6.1-2.

Let $\Gamma = \Gamma_1(N)$ for $N \geq 4$, so that Γ is torsion-free. Let $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\}$ denote the stabilizer of ∞ for the action of Γ on $\mathcal{H}$. Then the set of cusps S of $Y := \Gamma \backslash \mathbb{P}^1$ are in bijection with $\Gamma \backslash \operatorname{PSL}_2(\mathbb{Z})/\Gamma_\infty$. Any congruence subgroup of $\operatorname{SL}_2(\mathbb{Z})$ has finite index, so in particular S is finite.

For any cusp $s \in S$, there exists $\gamma \in \operatorname{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \infty = s$. For example, if $s = p/q$ as a rational fraction in lowest terms, take x, y such that $qy - px = 1$ and take

$$\gamma = \begin{pmatrix} q & x \\ p & y \end{pmatrix}.$$

We define the "distance from s" function $d_s(z) = \Im(\gamma^{-1}(z))^{-1}$. Since any two choices of γ differ by an element of $\pm\Gamma_\infty$, d is independent of the choice of γ. Using this distance function, we can define a "punctured ϵ-ball" $U_{s,\epsilon}$ around s, and we can take ϵ sufficiently small so that the $U_{s,\epsilon}$ for various $s \in S$ do not overlap.

Let $Y_0 = Y \setminus \bigcup_{s \in S} U_{s,\epsilon}$, which is compact. We may also compactify Y as X directly by adding in the cusp with an appropriate chart structure on the $U_{s,\epsilon} \cup \{s\}$.

Let R be a ring and M an $R[\Gamma]$-module. Let $\Gamma_s \subseteq \Gamma$ denote the stabilizer of a cusp s for the action of Γ on $\mathcal{H}$, and fix a generator $u_s \in \Gamma_s \simeq \mathbb{Z}$ of this stabilizer.

Definition 0.1. The *cuspidal/parabolic/Eichler* cohomology groups $H^1_P(\Gamma, M), H^2_P(\Gamma, M)$ is

$$H^1_P(\Gamma, M) = Z^1_P(\Gamma, M)/B^1(\Gamma, M)$$
$$H^2_P(\Gamma, M) = Z^2(\Gamma, M)/B^2_P(\Gamma, M)$$

where

$$C^1_P(\Gamma, M) = \bigcap_{s \in S} \{c : \Gamma \to M \,\big|\, c(u_s) \in (u_s - 1)M\}$$
$$Z^1_P(\Gamma, M) = C^1_P \cap Z^1(\Gamma, M)$$
$$B^2_P(\Gamma, M) = \partial C^1_P.$$

That is, $C_P^1(\Gamma, M)$ is the set of those cochains that become coboundaries when restricted to each Γ_s.

We have

$$H_P^1(\Gamma, M) := \ker\left(H^1(\Gamma, M) \to \prod_{s\in S} H^1(\Gamma_s, M)\right).$$

That is, Z_P^1 is given by those cocycles that become coboundaries when restricted to each Γ_s.

Intuition:

- $H^i(\Gamma, M)$ agrees with $H^i(Y, \underline{M})$, where $\underline{M}$ is the local system defined by $\Gamma\backslash(\mathcal{H} \times M)$.
- $H_P^i(\Gamma, M)$ can be identified with the image of $H_c^i(Y, \underline{M}) \to H^i(Y, \underline{M})$ of compactly supported cohomology.

X is some compact Riemann surface. Cut along the $2g$ fundamental loops of X to get a polygon with $4g$ sides; WLOG we may assume that all of the $U_{s,\epsilon}$ do not meet the loops. Fix one vertex q_0 of this polygon and make another cut from q_0 to each cusp s, and remove a small open circle around s. The cuts we have made pull back to get a fundamental domain for Γ on $\mathcal{H}$ except for a small neighborhood of the cusps. We triangulate this so that the boundaries of the preimages of the $U_{s,\epsilon}$ are each a 1-chain of this triangulation. We can do two things with this: first, by descending to the quotient, we get a simplicial complex structure on Y_0. We denote the set of i-simplices in this complex by S_i. Second, we can tessellate H_0 by Γ-translates of this triangulation to get a simplicial complex K for H_0. We let K_i denote the $R[\Gamma]$-module generated by the i-simplices in K—note that Γ preserves K, so we have a well-defined Γ-action on K. Two simplices lie in the same Γ-orbit if and only if they are identified in the quotient. Let $H^i(K, M)$ denote the cohomology of the cocomplex $C^i(K, M) = \mathrm{Hom}_{R[\Gamma]}(K_i, M)$.

Fact: The groups $H^i(K, M)$ agree with the group cohomology $H^i(\Gamma, M)$. Likewise the parabolic versions agree, where we take

$$H_P^1(K, M) = Z_P^1(K, M)/B^1(K, M), H_P^2(K, M) = Z^2(K, M)/B_P^2(K, M)$$

with

$$\begin{aligned} C_P^1(K, M) &= \bigcap_{s\in S}\{c : K_i \to M \big| c(\gamma_s) \in (u_s - 1)M\} \\ Z_P^1(K, M) &= C_P^1(K, M) \cap Z^1(K, M) \\ B_P^2(K, M) &= \partial C_P^1(K, M) \end{aligned}$$

where γ_s is a fundamental loop around the cusp s.

Proposition 0.2. *(Dimension formula) Suppose M is a finite dimensional vector space over a field R, and let g be the genus of X. Then*

$$\dim(H_P^1(\Gamma, M)) = (2g-2)\dim(M) + \dim(H^0(\Gamma, M)) + \dim(H_P^2(\Gamma, M)) + \sum_{s\in S}\dim((u_s - 1)M).$$

where u_s denotes a generator of $\Gamma_s \simeq \mathbb{Z}$.

Proof. Let $d = \dim M$ and $d' = \dim \bigoplus_{s\in S} M/(\gamma_s - 1)M$. Since $H^i(\Gamma, M) = H^i(K, M)$ and $H^i_P(\Gamma, M) = H^i_P(K, M)$, we will compute everything via the simiplicial complex K. For brevity of notation, we will just write H^i, H^i_P, C^i, C^i_P, etc.

- Letting Φ_0 be the fundmental domain for Y_0, we get a triangulation with

  $$\begin{aligned} 2g - 2 &= \#(S_2) - \#(S_1 - \{\gamma_s : s \in S\}) + \#(S_0) \\ &= \#S_2 - \#S_1 + \#S + \#S_0 \end{aligned}$$

  by Euler's formula, where we have to "add back in" the holes around the cusps to be able to consider the compact space X.

- We have $\#S_i = \mathrm{rank}_{R[\Gamma]}(K_i)$, and hence

  $$\dim C^i = \dim(\mathrm{Hom}_{R[\Gamma]}(K_i, M)) = \#(S_i) \dim_R(M).$$

- We have an exact sequence

  $$0 \to C^1_P \to C^1 \to \bigoplus_{s\in S} M/(\gamma_s - 1)M \to 0$$

  hence

  $$\begin{aligned} \dim_R(C^1_P) &= \#(S_1) \dim_R(M) - \sum_{s\in S} (\dim_R(M) - \dim_R((\gamma_s - 1)M)) \\ &= \#S_1 d - \#S d - d' \end{aligned}$$

- We have

  $$\begin{aligned} \dim H^0 &= \dim Z^0 = \#(S_0)d - \dim B^1 \\ \dim H^1_P &= \dim Z^1_P - \dim B^1 \\ \dim H^2_P &= \#(S_2)d - \dim B^2_P \\ &= \#(S_2)d - \#(S_1)d + \#Sd - d' + \dim Z^1_P \end{aligned}$$

  The third formula follows because

  $$\dim C^1_P = \dim C^1 - d'$$

  and $\dim B^2_P = \dim C^1_P - \dim Z^1_P$. Then we have

  $$\begin{aligned} \dim H^0 - \dim H^1_P + \dim H^2_P =& \#S_0 d - \dim B^1 - \dim Z^1_P + \dim B^1 \\ &+ \#S_2 d - (\#S_1 - \#S)d - d' + \dim Z^1_P \\ =& \#S_0 d - (\#S_1 - \#S)d - d' \\ =& (2g - 2)d - d'. \end{aligned}$$

  which is the formula we wanted.

■

Set $M = \mathrm{Sym}^n\mathbb{C}^2$, with Γ acting via the standard representation of SL_2. We can realize $M = \mathbb{C}[X,Y]$ with $\gamma \in \Gamma$ acting by sending the vector (X,Y) to $(X,Y)(\gamma^{-1})^T$.

Definition 0.3. We say a cusp s is *regular* if its stabilizer $\Gamma_s \subseteq \Gamma$ is Γ-conjugate to a subgroup of $\{u^h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\}$. Otherwise, Γ_s is conjugate to a subgroup of $\{-\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\}$ and we say that s is an irregular cusp.

Note that Γ_s is never conjugate to $\pm\Gamma_\infty$ because $-I \notin \Gamma$; the only two possibilities are either plus or minus, not both. Example: ∞ is always regular since we are assuming $-I \notin \Gamma$.

Proposition 0.4. *If Γ is torsion-free, then*

$$\dim_{\mathbb{C}} H^1_P(\Gamma, \mathrm{Sym}^n(\mathbb{C}^2)) = \begin{cases} (2g-2)(n+1) + n\#(S) + \delta\#S_{irr} : n > 0 \\ 2g : n = 0 \end{cases}$$

where S_{irr} denotes the set of irregular cusps and $\delta = 0$ or 1 matching the parity of n.

Remark. Let $S_n(\Gamma)$ denote the space of cusp forms of Γ of weight n. Then $\dim H^1_P(\Gamma, \mathrm{Sym}^n\mathbb{C}^2)$ agrees with $\dim S_{n+1}(\Gamma)$ via a direct comparison of the formulas. This alludes to the Eichler-Shimura isomorphism, and indicates why we might care about parabolic cohomology.

Proof. This will follow from the previous formula if we can show that

$$\dim H^0(\Gamma, M) = \dim H^2_P(\Gamma, M) = \begin{cases} 1 : n = 0 \\ 0 : n > 0 \end{cases}$$

$$\dim(\gamma_s - 1)M = \begin{cases} n : n \text{ odd and } s \text{ irregular} \\ n+1 : \text{else} \end{cases}$$

The first fact follows from the fact that $\mathrm{Sym}^n\mathbb{C}^2$ is irreducible as a Γ-module, so that it has no Γ-fixed elements if $n > 0$, implying the formula for H^0. The formula for H^2_P follows from the formula $H^2_P = M_\Gamma = 0$ (module of Γ-coinvariants), which we omit.

For the second formula, conjugacy gives an isomorphism $M/(u_s - 1)M \simeq M/(\pm u^h - 1)M$, with the sign based on whether s is regular or irregular. Then the map $M \to \mathbb{C} : P(X,Y) \mapsto P(1,0)$ is surjective with kernel $(u-1)M$. To see this, we note that matrices of the form $(\pm)\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ are precisely those that do not change the coefficient of the X^n term of $P(X,Y)$, where we allow the minus sign only if n is even. Otherwise, if n is odd, then one can show that the operator $-\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} + 1$ acts invertibly on M, so that $(u_s - 1)M = M$. We therefore get $\dim(u_s - 1)M = n$ in the first case and $\dim(u_s - 1)M = n + 1$ in the second case, which give the correct contributions toward the formula. ■