Worksheet #28: Midterm 3 Review
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Problem 1. Find the volume of the solid torus with boundary $\rho = \sin\varphi$. Torus means "donut-shaped"; for the given torus, how big is the "donut hole"?

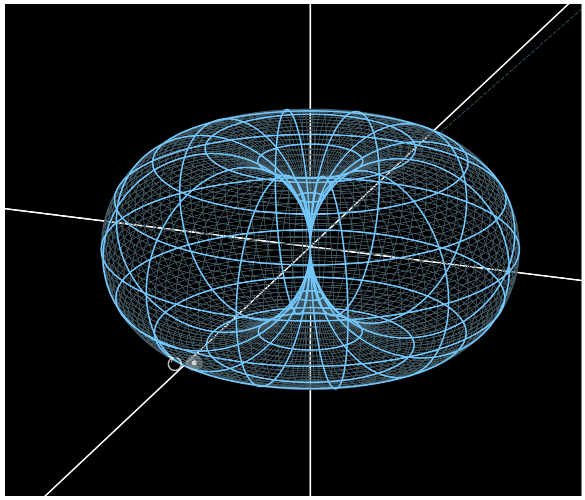
Using spherical coordinates, the bounds for this torus are

$$\begin{aligned} 0 \leq &\theta \leq 2\pi \\ 0 \leq &\varphi \leq \pi \\ 0 \leq &\rho \leq \sin\varphi \end{aligned}$$

Therefore the volume integral is

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi} \int_0^{\sin\varphi} 1 \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \frac{1}{3} \sin^3\varphi \cdot \sin\varphi \, d\varphi \, d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} \sin^4\varphi \, d\varphi \, d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} \left(\frac{1 - \cos(2\varphi)}{2} \right)^2 \, d\varphi \, d\theta \\ &= \frac{1}{12} \int_0^{2\pi} \int_0^{\pi} (1 - 2\cos(2\varphi) + \cos^2(2\varphi)) \, d\varphi \, d\theta \\ &= \frac{1}{12} \int_0^{2\pi} \int_0^{\pi} \left(1 - 2\cos(2\varphi) + \frac{1 + \cos(4\varphi)}{2} \right) \, d\varphi \, d\theta \\ &= \frac{1}{12} \int_0^{2\pi} \frac{3\pi}{2} \, d\theta \\ &= \frac{\pi^2}{4}. \end{aligned}$$

This "donut" isn't really a donut: its center is supposed to be at the z-axis, but it also touches the z-axis (when $\varphi = \rho = 0$). So there isn't actually a hole in the middle of this donut. It just pinches down to a point at the origin (see image).



Problem 2. Let C be the ellipse $x^2/4 + y^2 = 1$, oriented counterclockwise. Explain why Green's Theorem cannot be directly used to evaluate the line integral

$$\oint_C (y \log_4(x^2 + 4y^2) + 3x^2y^2 \cos(x^3))\, dx + (2y \sin(x^3))\, dy.$$

Find a way around this issue and evaluate the integral anyways. A hint is in this footnote.[1]

Green's Theorem cannot be applied immediately because this function is not defined on the interior of the ellipse: $\log(x^2 + 4y^2)$ is undefined at the origin. Instead, split up the integral as

$$\oint_C y \log_4(x^2 + 4y^2)\, dx + \oint_C 3x^2y^2 \cos(x^3)\, dx + 2y \sin(x^3)\, dy.$$

The vector field $\langle 3x^2y^2 \cos(x^3), 2y \sin(x^3)$ is conservative with potential function $y^2 \sin(x^3)$. Therefore, the second integral is 0. For the first integral, note that $x^2 + 4y^2 = 4$ on the ellipse, so the integrand simplifies to

$$\oint_C y \log_4(4)\, dx = \oint_C y\, dx.$$

We can finish integral in a few ways.

- Direct parametrization: We can parametrize the ellipse as $x = 2\cos t, y = \sin t$, $0 \le t \le 2\pi$, and the integral becomes

$$\begin{aligned}\int_0^{2\pi} \sin t \cdot (-2 \sin t)\, dt &= -2 \int_0^{2\pi} \frac{1 - \cos(2t)}{2}\, dt \\ &= -2\pi.\end{aligned}$$

- Green's Theorem: Let R be the (2-dimensional) interior of the ellipse. Then Green's Theorem tells us that

$$\oint_C y\, dx + 0\, dy = \iint_R -1\, dA.$$

[1]Break the line integral into two parts, one of which is a line integral of a conservative vector field and the other of which is easy to do directly.

So this integral is just the negative of the area of this ellipse. You could compute this area now—I'd reccomend stretching a circle using a change of variables—or you might have known the formula $A = ab\pi$ already, where a and b are the major and minor radii, respectively. The area of this ellipse is 2π, so we again conclude that line integral is -2π.

Thus the integral evaluates to -2π.

Problem 3. Let S be some region in the uv-plane with area 10; I'm not going to describe exactly what shape S is. Consider the change of variables

$$\begin{aligned} x &= 8u + 9v \\ y &= 11u + 12v \end{aligned}$$

Applying this change of variables transforms the region S to a new region T in the xy-plane.

(a) Prove that this change of variables is one-to-one (aka bijective) by giving an inverse change of variables.

(b) What is the area of T?

(a) what you have been given is the variables x, y each expressed in terms of u, v; what you're being asked to do is to express each of the variables u, v in terms of x, y. You can do this by thinking of the above change of variables as a linear system of equations and using the techniques you know from solving those. E.g. we can use substitution: the first equation gives $\frac{x-8u}{9} = v$, so the second equation tells us

$$y = 11u + 12\frac{x - 8u}{9} = \frac{4}{3}x - \frac{1}{3}u$$

which gives $u = -4x + 3y$. We sub this into the first equation to get $x = 8(-4x + 3y) + 9v$, which yields $v = \frac{11}{3}x - \frac{8}{3}y$. Therefore, the inverse change of variables is

$$\begin{aligned} u &= 4x - 3y \\ v &= \frac{11}{3}x - \frac{8}{3}y. \end{aligned}$$

(b) The Jacobian of the transformation is

$$J(u, v) = \begin{vmatrix} 8 & 9 \\ 11 & 12 \end{vmatrix} = 96 - 99 = -3.$$

Therefore the change of variables formula tells us

$$A_T = \iint_T 1\, dx\, dy = \iint_S |J(u, v)|\, du\, dv = \iint_S 3\, du\, dv = 3A_S.$$

That is, the area of T is 3 times the area of S, so the area of T is 30.

Problem 4. Let D be the disk of radius a centered at the origin. What's the average distance of a point on D from the origin?

The average distance from the origin can be computed as

$$\frac{\iint_D r \, dA}{A_D},$$

where A_D is the area of D, which is $a^2\pi$. We can do the double integral using polar coordinates: it is

$$\int_0^{2\pi} \int_0^a r^2 \, dr \, d\theta = \frac{2\pi}{3} a^3.$$

Dividing by the area, we conclude that the average distance from the center is $\frac{2}{3}a$.

Problem 5. Convert the following triple integral to Cartesian coordinates. You do not need to evaluate it.

$$\int_0^{\pi/2} \int_0^1 \int_{r^2}^{r} r^2 \cos\theta \, dz \, dr \, d\theta.$$

The surface $r = z$ is the cone $\sqrt{x^2+y^2} = z$, and the surface $r^2 = z$ is the elliptic paraboloid $x^2 + y^2 = z$. This gives bounds for z in terms of x and y. The bounds on r and θ correspond to integrating over the quarter circle of radius 1 in the first quadrant, so we can write bounds $0 \le x \le 1, 0 \le y \le \sqrt{1-x^2}$. When converting from cylindrical to Cartesian, we lose a factor of r in the integrand, and the leftovers are $r\cos\theta = x$. Therefore, the rectangular integral is

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} x \, dz \, dy \, dx.$$

Problem 6. Suppose a surface in 3D space is given by a level set $g(z, r, \theta) = 0$ in cylindrical coordinates. Derive the following formula for the normal vector to this surface:

$$\vec{n} = \left\langle g_r \cos\theta - \frac{1}{r} g_\theta \sin\theta, \; g_r \sin\theta + \frac{1}{r} g_\theta \cos\theta, \; g_z \right\rangle$$

We can treat each cylindrical coordinate z, r, θ as a function of rectangular coordinates x, y, z:

$$\begin{aligned} z &= z \\ r &= \sqrt{x^2+y^2} \\ \theta &= \arctan(y/x). \end{aligned}$$

We know that the normal vector of this surface is given by $\vec{n} = \langle g_x, g_y, g_z \rangle$, so we need to use the chain rule to express each othese in terms of the cylindrical coordinates.

$$\begin{aligned} g_x &= g_z \cdot \frac{\partial z}{\partial x} + g_r \cdot \frac{\partial r}{\partial x} + g_\theta \cdot \frac{\partial \theta}{\partial x} = 0 + g_r \cdot \frac{x}{\sqrt{x^2+y^2}} + g_\theta \cdot \frac{-y}{x^2+y^2} \\ g_y &= g_z \cdot \frac{\partial z}{\partial y} + g_r \cdot \frac{\partial r}{\partial y} + g_\theta \cdot \frac{\partial \theta}{\partial y} = 0 + g_r \cdot \frac{y}{\sqrt{x^2+y^2}} + g_\theta \cdot \frac{x}{x^2+y^2} \\ g_z &= g_z \cdot \frac{\partial z}{\partial z} + g_r \cdot \frac{\partial r}{\partial z} + g_\theta \cdot \frac{\partial \theta}{\partial z} = g_z \cdot 1. \end{aligned}$$

If we rewrite these results back in terms of cylindrical coordinates, we conclude that

$$
\begin{aligned}
g_x &= g_r \cos\theta - \frac{1}{r} g_\theta \sin\theta \\
g_y &= g_r \sin\theta + \frac{1}{r} g_\theta \cos\theta \\
g_z &= g_z
\end{aligned}
$$

which is the formula we were looking for.