Worksheet #27: Green-Eyed Monster
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Problem 1. Suppose that a cable has constant linear density k and has the shape of the helix

$$x = 4\cos t, y = 4\sin t, z = 3t, 0 \le t \le \pi/2.$$

Find its center of mass.

Let's compute the length form first:

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}\, dt \\ &= 5\, dt. \end{aligned}$$

Next, let's compute the mass of the cable:

$$m = \int_0^{\pi/2} k \cdot 5\, dt = \frac{5\pi}{2}k.$$

The coordinate of the center of mass are

$$\begin{aligned} \tilde{x} &= \frac{1}{m}\int_0^{\pi/2} 4k\cos t \cdot 5\, dt = \frac{8}{\pi} \\ \tilde{y} &= \frac{1}{m}\int_0^{\pi/2} 4k\sin t \cdot 5\, dt = \frac{8}{\pi} \\ \tilde{z} &= \frac{1}{m}\int_0^{\pi/2} 3k \cdot 5\, dt = \frac{3\pi}{4}. \end{aligned}$$

Therefore the center of mass is $(8/\pi, 8/\pi, 3\pi/4)$.

I will discuss Green's Theorem in anticipation of tomorrow's lecture, and the midterm.

Problem 2. Use Green's Theorem to give an alternative proof that the line integral of a conservative vector field around a loop is 0.

A conservative vector field is a vector field of the form $\nabla f = \langle f_x, f_y$ for some (smooth) scalar function f. Consider a loop γ, and let R be the region that it encloses. Green's Theorem states that

$$\oint_\gamma \nabla f \cdot d\vec{r} = \pm \iint_R (f_{yx} - f_{xy}) dA,$$

where the sign of the second integral depends on whether γ run counterclockwise or clockwise. But by Clairaut's Theorem $f_{yx} = f_{xy}$, so the second integral is just 0.

Problem 3. Let γ be the path that travels clockwise around the perimeter of the trapezoid with vertices $(-1,-1), (1,-1), (1,2), (-1,4)$. Evaluate the line integral

$$\oint_\gamma \langle x^3 - yx, 6y - 9x\rangle \cdot d\vec{r}$$

(Numerical answer $2 \cdot 3^{-1} \cdot 109$.)

We can describe the trapezoid with bounds

$$\begin{aligned} -1 \leq& x \leq 1 \\ -1 \leq& y \leq 3 - x \end{aligned}$$

The scalar curl of the vector field $\langle x^3 - yx, 6y - 9x \rangle$ is $\frac{\partial}{\partial x}(6y - 9x) - \frac{\partial}{\partial y}(x^3 - yx) = -9 + x$. It's important to note that γ travels *clockwise*, which means that we need to add a minus sign in front of the double integral when applying Green's Theorem:

$$\begin{aligned} \oint_\gamma \langle x^3 - yx, 6y - 9x \rangle \cdot d\vec{r} &= -\int_{-1}^{1} \int_{-1}^{3-x} (x - 9)\, dy\, dx \\ &= \int_{-1}^{1} \int_{-1}^{3-x} 9 - x\; dy\, dx \\ &= \int_{-1}^{1} (9 - x)(3 - x + 1)\, dx \\ &= \int_{-1}^{1} x^2 - 13x + 36\; dx \\ &= \left[\frac{1}{3}x^3 - \frac{13}{2}x^2 + 36x \right]_{-1}^{1} \\ &= \frac{2}{3} + 72 \\ &= \frac{218}{3}. \end{aligned}$$