Worksheet #26: Whose Line Integral Is It Anyway?
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Math 53: Fall 2022
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Problem 1. Sketch the following vector fields, and then determine whether or not the given vector field is conservative.

If you think the vector field is conservative:

- Find a potential function for it.
- Compute the path integral of this vector field between the points $(1, 1)$ and $(3, 2)$.

If you think that the vector field is not conservative:

- Prove it by showing that the vector field isn't irrotational.
- Using your sketch, draw a loop over which the path integral looks nonzero.

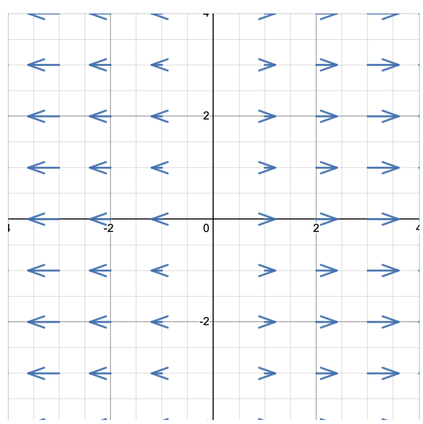
(a) $F(x, y) = \langle x, 0\rangle$

(b) $F(x, y) = \langle xy, xy\rangle$

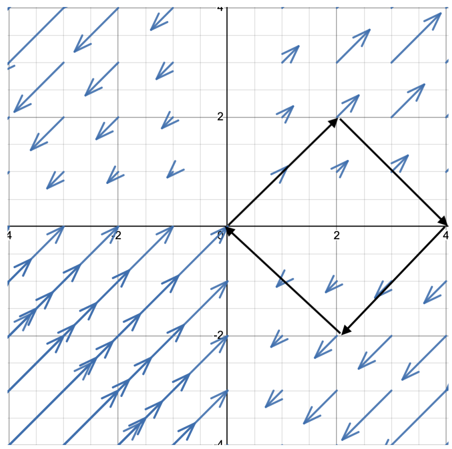
(c) $F(x, y) = \langle \sin(y), \cos(x)\rangle$

(d) $F(x, y) = \frac{1}{x^2+y^2}\langle x, y\rangle$

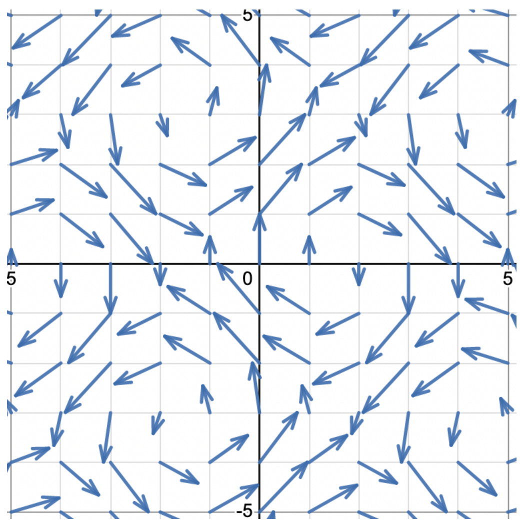
(a) This vector field is convervative. Its potential function is $f(x, y) = \frac{x^2}{2} + C$. By the Fundamental Theorem for Line Integrals, the line integral along *any* path from $(1, 1)$ to $(3, 2)$ is $f(3, 2) - f(1, 1) = 4$. Sketch:



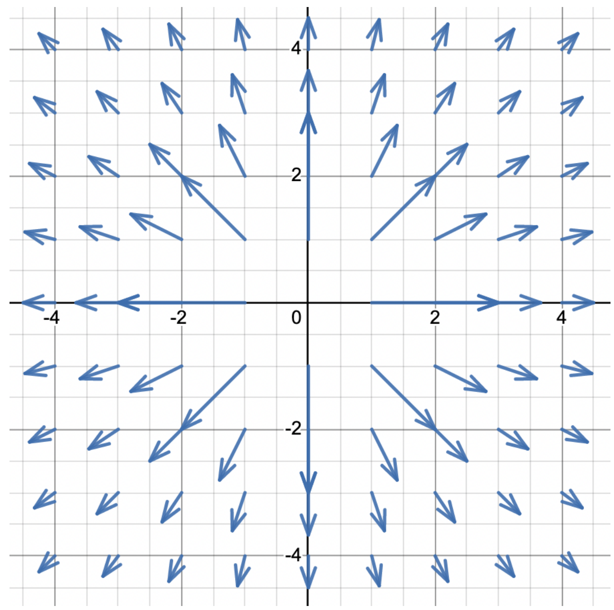
(b) This vector field is not conservative; its scalar curl is $y - x$, which is not always zero. The square path in black on the vector field has positive path integral.



(c) This is also not conservative, since its scalar curl is $-\sin(x) - \cos(y) \neq 0$. You can visibly see the swirliness in graph of vector field below; going around any of the "swirls" will give a nonzero path integral.



(d) This vector field is conservative: it is the gradient of the function $f(x, y) = \frac{1}{2}\ln(x^2 + y^2) + C$. Note that it is *not* enough to verify that the vector field is irrotational in order to prove that it is conservative in this case because the domain is not *simply connected* (it is missing the origin). You need to actually exhibit a scalar function whose gradient is the vector field in this situation. By the Fundamental Theorem, the path integral of this vector field over any path from $(1, 1)$ to $(3, 2)$ is $\frac{1}{2}\ln(3^2 + 2^2) - \frac{1}{2}\ln(1 + 1) = \frac{1}{2}\ln(13/2)$.



Problem 2. Suppose that a cable has constant linear density k and has the shape of the helix

$$x = 4\cos t, y = 4\sin t, z = 3t, 0 \leq t \leq \pi/2.$$

Find its center of mass.

This will be on the solutions for the next worksheet (titled "Green") on the website).

Problem 3.

(a) The vector field $F = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z\rangle$ is conservative. Find a potential function for it.

(b) Let C be the line segment starting at $(0, 0, 2)$ and ending at $(0, 3, 0)$. Evaluate

$$\int_C \langle 3x^2yz - 3y + e^{\pi \arctan(yz)^2}, x^3z - 3x + 2\cos(x^2), x^3y + 2z\rangle \cdot dr.$$

(Hint: break the integral into two integrals, and use part (a) for one of these.)

(a) $f(x, y, z) = x^3yz - 3xy + z^2$ is a potential function for this vector field (i.e. the gradient of this function is the given vector field). For a systematic method of finding potential functions of a given vector field, review the solutions for the "Partials 2" worksheet.

(b) We split the integral into

$$\int_C \nabla f \; d\vec{r} + \int_C \langle e^{\pi \arctan(yz)^2}, 2\cos(x^2), 0\rangle \; d\vec{r},$$

where f is the potential function from part (a). By the Fundamental Theorem for Line Integrals, the first integral is $f(0, 3, 0) - f(0, 0, 2) = -4$. For the second integral, we can parametrize the line segment linearly as $\vec{r}(t) = (0, 3t, 2 - 2t), 0 \leq t \leq 1$. Thus, the second integral is

$$\int_0^1 \langle e^{\pi \arctan(yz)^2}, 2\cos(0^2), 0\rangle \cdot \langle 0, 3, -2\rangle \; dt = \int_0^1 6 \; dt = 6,$$

noting that $x = 0$ everywhere on the line segment. Adding these two integrals together, the final answer is 2.