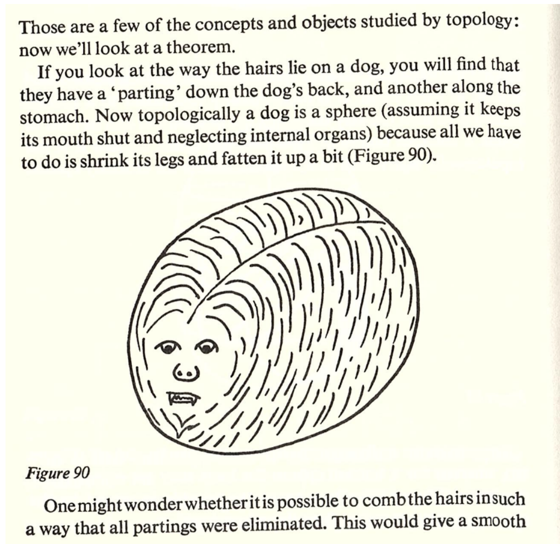
Worksheet #25: The Path of Pain
Date: 11/02/2022
Math 53: Fall 2022
Instructor: Norman Sheu
Section Leader: CJ Dowd

Problem 0. I will discuss the Hairy Ball Theorem. Yes, this is its real name.



Those are a few of the concepts and objects studied by topology: now we'll look at a theorem.

If you look at the way the hairs lie on a dog, you will find that they have a 'parting' down the dog's back, and another along the stomach. Now topologically a dog is a sphere (assuming it keeps its mouth shut and neglecting internal organs) because all we have to do is shrink its legs and fatten it up a bit (Figure 90).

Figure 90

One might wonder whether it is possible to comb the hairs in such a way that all partings were eliminated. This would give a smooth

Problem 1.

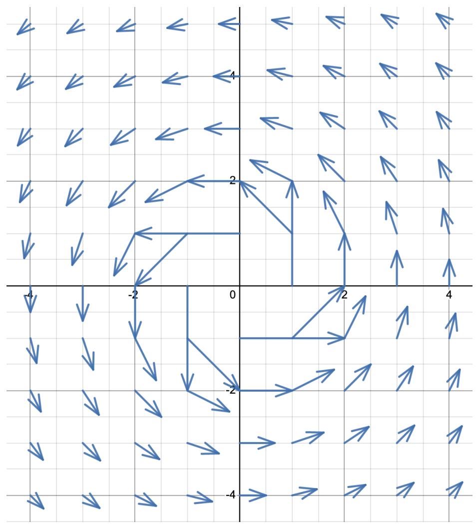
(a) Sketch the gradient vector field of $f(x, y) = \arctan(y/x)$. Note that this function isn't defined when $x = 0$, but despite this you can extend the gradient field to a smooth vector field on all of $\mathbb{R}^2 \setminus \{0\}$.

(b) Let $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ be the extended vector field from part (a). Let γ be the path traveling around the unit circle, starting and ending at $(1, 0)$. Compute

$$\int_\gamma F \cdot d\vec{r}.$$

Is F conservative? Does this contradict anything?

(a) The gradient of this vector field is $\nabla f = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$, for $x \neq 0$ (and undefined when $x = 0$). The vectors always point counterclockwise in a circle around the origin, and their lengths are inversely proportional to the distance to the origin. Here's Desmos's sketch of this vector field:



(b) We parametrize γ as $\vec{r}(t) = \langle \cos t, \sin t \rangle$. Then the path integral is

$$\int_0^{2\pi} \left\langle -\frac{\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right\rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.$$

The path integral of any conservative vector field over a loop is 0, so we conclude that $F = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ is *not* conservative. The reason this doesn't contradict $\nabla f = F$ is because it isn't quite true that $\nabla f = F$—even though they agree everywhere they are defined, the domain of ∇f excludes the y-axis, but F is defined everywhere except the origin. In fact, we can't even define the given path integral over ∇f, since it crosses the y-axis. The failure of ∇f to be defined globally ultimately reflects the fact that there isn't a single function defined everywhere on $\mathbb{R}^2 \setminus \{0\}$ whose gradient is $\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$; somehow the "global" picture has an impediment that the "local" picture does not see.

Problem 2. Let γ be the path (t, t^2), starting at $(0,0)$ and ending at $(2,4)$.

(a) Evaluate

$$\int_\gamma x \, ds,$$

where ds denotes the length form. How does the value of this integral change if the path γ is replaced by its reverse path?

(b) Evaluate

$$\int_\gamma x \, dy.$$

How does the value of this integral change if the path γ is replaced by its reverse path?

(a) The length form is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{1 + 4t^2} dt.$$

The path starts at $t = 0$ and ends at $t = 2$, and $x = t$, so the path integral is

$$\int_0^2 t\sqrt{1+4t^2}\, dt.$$

To evaluate this, make the substitution $u = 1 + 4t^2, du = 8t\, dt$. Then the integral becomes

$$\frac{1}{8}\int_0^{17} \sqrt{u}\, du = \frac{2}{24}[(17)^{3/2} - 1] = \frac{17\sqrt{17}-1}{12}.$$

The length form ds is *orientation-free*; it does not detect whether the curve is going forward or backward. (The reason why simply flipping the bounds does not give the negative answer is because when we first derived the formula for ds in terms of dt, we assumed that the lower bound was less than the upper bound because we took some absolute values at some point.) Therefore, tracing the curve backwards would give the same answer.

(b) In terms of t, we have $dy = 2t\, dt$. Therefore, the integral is

$$\int_0^2 t \cdot 2t\, dt = \frac{16}{3}.$$

This time, the form dy *does* care about orientation; going backwards along the curve would give us the negative of what we just got.