Worksheet #25: The Path of Pain
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Problem 0. I will discuss the Hairy Ball Theorem. Yes, this is its real name.

Those are a few of the concepts and objects studied by topology: now we'll look at a theorem.

If you look at the way the hairs lie on a dog, you will find that they have a 'parting' down the dog's back, and another along the stomach. Now topologically a dog is a sphere (assuming it keeps its mouth shut and neglecting internal organs) because all we have to do is shrink its legs and fatten it up a bit (Figure 90).

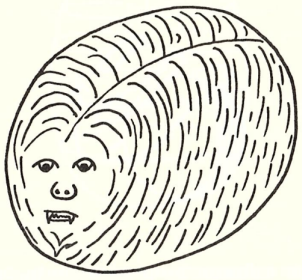

Figure 90

One might wonder whether it is possible to comb the hairs in such a way that all partings were eliminated. This would give a smooth

Problem 1.

(a) Sketch the gradient vector field of $f(x, y) = \arctan(y/x)$. Note that this function isn't defined when $x = 0$, but despite this you can extend the gradient field to a smooth vector field on all of $\mathbb{R}^2 \setminus \{0\}$.

(b) Let $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ be the extended vector field from part (a). Let γ be the path traveling around the unit circle, starting and ending at $(1, 0)$. Compute

$$\int_\gamma F \cdot d\vec{r}.$$

Is F conservative? Does this contradict anything?

Problem 2. Let γ be the path (t, t^2), starting at $(0, 0)$ and ending at $(2, 4)$.

(a) Evaluate

$$\int_\gamma x \, ds,$$

where ds denotes the length form. How does the value of this integral change if the path γ is replaced by its reverse path?

(b) Evaluate

$$\int_\gamma x \, dy.$$

How does the value of this integral change if the path γ is replaced by its reverse path?