Worksheet #24: Everybody Loves the Jacobian
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Problem 1. Give a relatively conceptual explanation of why the surface area form is

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

in the case of a surface of the form $z = f(x, y)$. (Think about parallelograms.)

This should represent the area of the parallelogram on the surface lying over a small square on the xy-plane. The sides of this paralellogram are given by the vectors

$$\langle 1, 0, \frac{\partial z}{\partial x}\rangle, \langle 0, 1, \frac{\partial z}{\partial y}\rangle$$

The area of a parallelogram spanned by vectors $\vec{u}, \vec{v}$ is $\|\vec{u} \times \vec{v}\|$. In this case, this is

$$\|\langle 1, 0, \frac{\partial z}{\partial x}\rangle \times \langle 0, 1, \frac{\partial z}{\partial y}\rangle\| = \|\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\rangle\| = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

which is exactly what we expect.

Problem 2. Let u, v be variables and θ a constant. Describe in words what the change of variables

$$\begin{aligned} x &= u\cos\theta - v\sin\theta \\ y &= u\sin\theta + v\cos\theta \end{aligned}$$

does. (Hint: set $u = r\cos\varphi$ and $v = r\sin\varphi$ and apply some trig identities. Drawing a picture might help.) Compute the Jacobian of this transformation and explain why your answer is reasonable.

If we express u and v in polar coordinates as in the hint, we have

$$\begin{aligned} x &= r\left(\cos\varphi\cos\theta - \sin\varphi\sin\theta\right) \\ y &= r\left(\cos\varphi\sin\theta + \sin\varphi\cos\theta\right) \end{aligned}$$

Recalling the angle addition formulas from trigonometry, the terms in parentheses are $\cos(\varphi+\theta)$ and $\sin(\varphi+\theta)$. This means that the polar coordinates of the point (x, y) is $(r, \theta+\varphi)$. That is, the radius is the same as it was for (u, v), and the angle has changed by θ. We conclude that this change of variables represents the transformation given by rotation by the angle θ.

We compute the Jacobian as

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1.$$

This is saying that area of any region in the old coordinates is the same as the area of that region in the new coordinates. This makes sense in this case because rotation is a rigid transformation: it never stretches or compresses anything.

Problem 3. (Stewart Exercise 15.9.22) By applying an appropriate change of variables, compute the area enclosed by the four curves $xy = a$, $xy = b$, $xy^{7/5} = c$, and $xy^{7/5} = d$, where $0 < a < b$ and

$0 < c < d$, and contained in the first quadrant. This computation is important in thermodynamics, since this area represents the work done by an ideal Carnot engine.

It doesn't look pleasant to give bounds on this region in terms of x and y, so we'll change variables to make the region nicer. Setting $u = xy$ and $v = xy^{7/5}$ seems like a reasonable choice, since the bounds are $a \leq u \leq b, c \leq v \leq d$, i.e. a rectangle. However, this comes at the price of computing the Jacobian.

There are two ways to compute this Jacobian. The first is direct: we have to express x and y in terms of u and v. If you play around with it, you'll find that we have

$$\begin{aligned} x &= (u^{7/5}/v)^{5/2} = u^{7/2}v^{-5/2} \\ y &= (v/u)^{5/2} = u^{-5/2}v^{5/2} \end{aligned}$$

whence the Jacobian is

$$\begin{vmatrix} \frac{7}{2}u^{5/2}v^{-5/2} & -\frac{5}{2}u^{7/2}v^{-7/2} \\ -\frac{5}{2}u^{-7/2}v^{5/2} & \frac{5}{2}u^{-5/2}v^{3/2} \end{vmatrix} = \frac{5}{2v}.$$

But this is slightly unpleasant to do, so I'll illustrate another way. The Jacobian of the *inverse* change of variables, i.e. changing from x, y to u, v is simpler to compute: it is

$$\begin{vmatrix} y & x \\ y^{7/5} & \frac{7}{5}xy^{2/5} \end{vmatrix} = \frac{2}{5}xy^{7/5}.$$

But the Jacobian of the change of variables in the forward direction should be the reciprocal of this, since applying both changes of variables in sequence gives the identity change of variables. Under our interpretation of the Jacobian as a measure of how the volume of a box scales, we conclude that the Jacobians have to be reciprocals at every point. (If this doesn't make sense, don't worry too much about it, since the proper setting for this is a better treatment of the Jacobian matrix and the chain rule. Just think about the first way to compute the Jacobian instead.) The reciprocal is $\frac{5}{2xy^{7/5}} = \frac{5}{2v}$, which agrees with our previous answer.

Now we need to actually integrate: the area is

$$\int_a^b \int_c^d \frac{5}{2v}\, dv\, du = \frac{5}{2}(b-a)\left(\ln(d) - \ln(c)\right).$$