Worksheet #23: Pondering The Orb
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Problem 1. Convert

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dx\, dy$$

into spherical coordinates and then evaluate the integral. As always, try to sketch the region of integration.

The innermost integral tells us that we integrating over a region lying above the cone $z = \sqrt{x^2+y^2}$ and below the hemisphere $z = \sqrt{18-x^2-y^2}$. This region looks sort of like a snowcone The outer two integrals tell us that we are only considering the portion of this volume contained in the quarter cylinder $x^2+y^2 = 9$ and lying above the positive quadrant. If we solve for the intersection of the cone and the sphere by setting $\sqrt{18-x^2-y^2} = \sqrt{x^2+y^2}$ we find that this intersection is the circle $r = 3$, $z = 3$, which lies exactly on the cylinder, so the only actual information that the outer bounds tell us is that we are integrating exactly over the potion of the snowcone lying above the positive quadrant.

The radius ρ always ranges from 0 to the sphere, which has radius $\sqrt{18}$. Since we are integrating exactly over the positive quadrant, we have θ rangning from 0 to $\pi/4$. Finally, φ ranges from 0 to the angle that the side cone makes with the z-axis. Since the the side of the cone has slope 1, we can determine this angle as $\pi/4$. Therefore, remembering the Jacobian and noting that $x^2+y^2+z^2 = \rho^2$, we can rewrite the given integral as

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{18}} \rho^4 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

If you do this integral, you get $\frac{486\sqrt{2}\pi}{5}$. Spherical coordinates are nice here because all the bounds are constants; the weird snowcone thing is in fact the nicest possible region to integrate over because it's the spherical coordinates analogue of a box.

Problem 2. Consider a right circular cone with constant unit density, radius a, and height h.

(a) Find bounds that describe such a cone in cylindrical coordinates.

(b) Find the moment of inertia of the cone about its axis of symmetry.

(c) Find the moment of inertia of the cone about a diameter of its base.

(a) I'll take the cone whose vertex is at the origin and whose base lies at height $z = H$ (so it's upside down). The equation for such a cone is $z = \frac{H}{a}r$; one way to see this is to note that I want the height to be H with the radius is a, and the height scales linearly with the radius, so this must be the correct eqaution. Therefore this cone is given by the following bounds in sphereical coordinates:

$$\frac{H}{a}r \le z \le H$$
$$0 \le r \le a$$
$$0 \le \theta \le 2\pi$$

(b) The axis of rotation here is the z-axis, so we need to determine the distance of a point from the z-axis. But this is just the radius r. Therefore, using the formula for moment of inertia and remembering the extra factor of r from the Jacobian in cylindrical coordinates, the MOI is

$$\int_0^{2\pi}\int_0^a\int_{\frac{H}{a}r}^H r^3\,dz\,dr\,d\theta.$$

Let's evaluate this:

$$\begin{aligned}
&= \int_0^{2\pi}\int_0^a Hr^3 - \frac{H}{a}r^4\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{Ha^4}{4} - \frac{Ha^5}{5a}\,d\theta \\
&= \frac{\pi H a^4}{10}.
\end{aligned}$$

If you look up the MOI of a cone, you'll find that the formula is usually given as $3ma^2/10$, where m is the mass of the cone. This agrees with our answer, since our cone has volume = mass $\frac{1}{3}\pi H a^2$.

(c) If we instead take the axis of rotation to be the line $z = H, y = 0$, i.e. the diameter of the base of the cone parallel to the x-axis, then the distance of a point to this line is $\sqrt{(H-z)^2 + y^2}$. We can rewrite this in cylindrical coodinates as $\sqrt{(H-z)^2 + r^2\sin^2\theta}$. Plugging this into the MOI formula, again remembering the Jacobian r, the MOI is

$$\int_0^{2\pi}\int_0^a\int_{\frac{H}{a}r}^H r(H-z)^2 + r^3\sin^2\theta\,dz\,dr\,d\theta.$$

The integration computation is kinda messy, but certainly doable within elementary means. Here's how I did it:

$$\begin{aligned}
&= \int_0^{2\pi}\int_0^a -\frac{1}{3}r\left[(H-H)^3 - (H-Hr/a)^3\right] + [H - Hr/a]r^3\sin^2\theta\,dr\,d\theta \\
&= \frac{H^3}{3}\int_0^{2\pi}\int_0^a -\frac{r^4}{a^3} + \frac{3r^3}{a^2} - \frac{3r^2}{a} + r\,dr\,d\theta + H\int_0^{2\pi}\int_0^a [r^3 - \frac{r^4}{a}]\sin^2\theta\,dr\,d\theta \\
&= \frac{H^3a^2}{3}\int_0^{2\pi}\left(\frac{-1}{5} + \frac{3}{4} - \frac{3}{3} + \frac{1}{2}\right)d\theta + Ha^4\int_0^{2\pi}\left(\frac{1}{4} - \frac{1}{5}\right)\sin^2\theta d\theta \\
&= \frac{H^3a^2\pi}{30} + \frac{Ha^4\pi}{20}.
\end{aligned}$$

(It's possible I've made a mistake when doing this, so do this computation yourself and compare. If you get something different, check to see where our answers diverge. If you think that I made a mistake there, let me know!)