Worksheet #22: Thrice is Nice
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Problem 1. Consider the 3D region enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$. Sketch this region (you saw it in lecture on Tuesday). Then set up *six* different integrals to compute the volume of this region corresponding to the six possible orders of integration (i.e. the six permutations of dx, dy, dz). No need to evaluate these.

Here are the six integrals:

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx$$

$$\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{1-y} dz\,dx\,dy$$

$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} dy\,dz\,dx$$

$$\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$$

$$\int_{0}^{1}\int_{0}^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$$

$$\int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz.$$

The way to do this systematically is as follows: say we want to do the integration order $dt\,du\,dv$, where t, u, v are some permutation of x, y, z. For each fixed value u_0, v_0 of u and v, we determine the minimum and maximum values of t such that the point (t, u_0, v_0) lie in the region. This maximum/minimum will likely depend on the choice of u_0 and v_0—that is, these bounds will be some *function* of u_0 and v_0. These are the bounds that go on the innermost integral, the one that integrates dt. Next, for a fixed value $v_0 = v$ of v, we determine the minimum and maximum values of u such that (t, u, v_0) lies in the region (letting t be anything now). Again, this minimum and maximum value of u will be a function that depends on v_0; these are the bounds of the second integral, which integrates du. Finally, we determine the minimum and maximum value of v over the entire region (allowing any value of t and u). These bounds should be absolute constants, not depending on any other variable, and they form the bounds of the outermost integral over dv.

But explaining this in text format probably isn't enough to give good intuition on what you are doing. It's best to do a bunch of examples and then come back to this to figure out what you're doing and why it works to compute the triple integral.