Worksheet #21: This is getting out of hand! Now, there are two of them!
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Instructor: Norman Sheu
Section Leader: CJ Dowd

Problem 1. Let R be the rectangle with corners $(0,0),(2,0),(2,3),(0,3)$. Suppose that $f(x,y)$ is a function such that $f_x > 0$ and $f_y < 0$ on R. If we want to use a Riemann sum to overestimate the integral $\int\int_R f(x,y)\,dx\,dy$, where we break up R into six 1×1 squares, where should we pick the sample points?

The sample points should all go in the lower right corners of each square, since this is where the function will be at its maximum on each square.

Problem 2. You are asked to integrate some function $f(x,y)$ over some region R in the plane. In each of the following scenarios, say whether you are inclined to use the integration order $dx\,dy$ or the integration order $dy\,dx$. How strongly do you feel about your chosen order, i.e. is the other order still potentially worth trying? Except for part (d), set up a double integral with your chosen order. If you think both orders are plausible, set up a double integral for both orders.

(a) The region R is the triangle with corners $(0,0),(1,1),(1,-1)$.

(b) The region R is the region defined by $x^2 \le y \le x$.

(c) The region R is the bounded region between the two curves $x = y - y^3$ and $x = y^2 - 1$.

(d) The function is $f(x,y) = e^{x^2}$.

(a) Both are reasonable, but maybe try $dy\,dx$ first since you'll eventually have to do two integrals for the other order. The integrals are

$$\int_0^1 \int_{-x}^{x} f(x,y)\,dy\,dx$$

$$\int_{-1}^{1}\int_{|y|}^{1} f(x,y)\,dx\,dy = \int_{-1}^{0}\int_{-y}^{1} f(x,y)\,dx\,dy + \int_0^1\int_y^1 f(x,y)\,dx\,dy$$

(b) Both are fine. The integrals are

$$\int_0^1 \int_{x^2}^{x} f(x,y)\,dy\,dx$$

$$\int_0^1 \int_y^{\sqrt{y}} f(x,y)\,dx\,dy.$$

(c) Only the $dx\,dy$ order seems reasonable. Doing the other order would likely involve inverting the function $x = y - y^3$, which is very difficult.

$$\int_{-1}^{1}\int_{y^2-1}^{y-y^3} f(x,y)\,dx\,dy.$$

(The outer bounds come from solving $y - y^3 = y^2 - 1$, and the only solutions are ± 1. To make sure that the inner bounds are in the right order, note that the upper bound will always be higher than the lower bounds, and that $y - y^3 > y^2 - 1$ at $y = 0$.)

(d) If either order of integration works, it's almost certainly the $dy\,dx$ order, since $\int e^{x^2}\,dx$ cannot be expressed in terms of elementary functions. At least with the $dy\,dx$ order, it's possible that some additional factor will appear to make the function integrable. For example, if the region is $0 \leq y \leq x$, $0 \leq x \leq 1$, then we have

$$\int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 xe^{x^2}\,dx = \frac{1}{2}(e-1).$$

Problem 3. The value of the integral

$$\int\int_R (3 - x^2 + 2x - 4y^2)\,dx\,dy$$

depends on the region R chosen. (It might even be undefined if R is unbounded.) What choice of region R maximizes the value of this integral?

The region R should be chosen to include all points where the integrand is positive, since this points contribute positively towards the integral, and exclude all the points where the integrand is negative, since these contribute negatively toward the integral. We'll include the points where the integrand vanishes as well, which doesn't affect the integral. This means that the region R where the integral is maximized is given by the inequality

$$3 - x^2 + 2x - 4y^2 \geq 0,$$

which we can rewrite in the standard form for an ellipse as

$$\frac{(x-1)}{4} + y^2 \leq 1.$$