Worksheet #20: Prepare for Trouble//And Make It Double
Date: 10/21/2022
Math 53: Fall 2022
Instructor: Norman Sheu
Section Leader: CJ Dowd

Problem 1. Let D be the region in the first quadrant that lies between the circles $x^2+y^2=1$ nad $x^2+y^2=2$. Sketch this region, and evaluate

$$\int\int_D x\,dA.$$

(Polar coordinates is probably the way to go.) Determine the center of mass of D, assuming it has uniform density.

In polar coordinates, this region has bounds $0 \le \theta \le \pi/2, 1 \le r \le \sqrt{2}$, so the integral is

$$\int_0^{\pi/2}\int_1^{\sqrt{2}} xr\,dr\,d\theta = \int_0^{\pi/2}\int_1^{\sqrt{2}} r^2\cos\theta\,dr\,d\theta$$

Remember the extra factor of r! This integral is

$$\int_0^{\pi/2} \frac{r^3}{3}\cos\theta\Big|_{r=1}^{\sqrt{2}}\,d\theta = \frac{1}{3}(2\sqrt{2}-1)\int_0^{\pi/2}\cos\theta dt = \frac{1}{3}(2\sqrt{2}-1).$$

Since scaling the mass of an object does not change its center of mass, we can assume that the region has uniform density 1, so its total mass is equal its area, which is $\frac{\pi}{4}$ (you can compute this by subtracting the area of the two circular sectors). To get the x-coordinate of the center of mass, we divide the number we got above by the total mass, so we conclude $\tilde{x} = \frac{4}{3\pi}(2\sqrt{2}-1)$. By symmetry, $\tilde{y}$ is the same number, so the center of mass is the point

$$(\frac{4}{3\pi}(2\sqrt{2}-1), \frac{4}{3\pi}(2\sqrt{2}-1)) \approx (0.776, 0.776).$$

Problem 2. Sketch the region $0 \le x \le 1, x^2 \le y \le 1$ in the plane. Evaluate the double integral

$$\int_0^1\int_{x^2}^1 \sqrt{y}\sin y\,dy\,dx.$$

(Hint: Reverse the order of integration.)

We define alternative bounds for this region: $0 \le y \le 1, 0 \le x \le \sqrt{y}$. By Fubini's theorem, we can interchange the order of integration, so the above integral is equal to

$$\int_0^1\int_0^{\sqrt{y}} \sqrt{y}\sin y\,dx\,dy = \int_0^1 y\sin y\,dx.$$

Now we can evaluate this by integration by parts:

$$\begin{aligned}\int_0^1 y\sin y\,dx &= -y\cos y\Big|_0^1 + \int_0^1 \cos y dt \\ &= -\cos(1)+\sin(1).\end{aligned}$$

Problem 3. Find the moment of inertia of:

(a) A square with side length a and constant mass density ρ, rotating about its center.

(b) A disk with radius 1 and constant mass density ρ, rotating in 3D space about its diameter. (You'll probably need to use a reverse trig substitution. You can think of this as the MOI associated to flipping a coin.)

(a) The square distance to the origin is $x^2 + y^2$, so the MOI is

$$\begin{aligned}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}(x^2+y^2)\rho\, dy\, dx &= \rho\int_{-a/2}^{a/2} ax^2 + \frac{a^3}{12}\, dx\\ &= \rho\left(\frac{a^4}{12}+\frac{a^4}{12}\right) = \frac{\rho a^4}{6}.\end{aligned}$$

(b) We can take the disk to be the unit disk and the axis of rotation to be the x-axis. The distance from the x-axis is given by the y-coordinate, so the MOI is

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^2\rho\, dy\, dx = \frac{2\rho}{3}\int_{-1}^{1}(1-x^2)^{3/2}\, dx.$$

Make the reverse trig substitution $x = \cos\theta, dx = -\sin\theta d\theta$:

$$\begin{aligned}&= \frac{2\rho}{3}\int_{\pi}^{0}(1-\cos^2\theta)^{3/2}(-\sin\theta)\, d\theta\\ &= \frac{2\rho}{3}\int_{0}^{\pi}\sin^4\theta d\theta,\end{aligned}$$

where we use the minus sign to flip the bounds of integration and we note that $\sin\theta \geq 0$ on $0 \leq \theta \leq \pi$, so no absolute value bars are needed when simplifying the square root,

$$\begin{aligned}&= \frac{2\rho}{3}\int_{0}^{\pi}\left(\frac{1-\cos(2\theta)}{2}\right)^2 d\theta\\ &= \frac{\rho}{6}\int_{0}^{\pi} 1 - 2\cos(2\theta) + \cos^2(2\theta) d\theta\\ &= \frac{\rho}{6}\left[\frac{3}{2}\theta - \sin(2\theta) + \frac{1}{8}\sin(4\theta)\right]_0^{\pi}\\ &= \frac{\pi\rho}{4}.\end{aligned}$$