**Worksheet #19: Midterm 2 Review, Part 2**
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**Math 53: Fall 2022**
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**Problem 1.** For each of the following properties, either give an example of a continuously twice-differentiable function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ satisfying the property or prove that no such function exists.

(a) $f_{xx}f_{yy} - (f_{xy})^2 = 0$ at $(x, y) = (0, 0)$ and $f$ has a local maximum at $(0, 0)$. Alternatively, something like $f(x, y) = -x^2$ or $f(x, y) = -y^2$ works, where the function is independent of one of the variables.

(b) $f_{xy} = 0$ at $(0, 0)$ and $f$ has a saddle point at $(0, 0)$.

(c) $f$ has no maximum when rescricted to the unit circle $x^2 + y^2 = 1$.

(d) $f$ has infinitely many critical points when restricted to the unit circle $x^2 + y^2 = 1$.

(a) $f(x, y) = -x^4y^4$ satisfies this.

(b) $f(x, y) = x^2 - y^2$ satisfies this.

(c) This is impossible by the Extreme Value Theorem: the circle is closed and bounded, so any continuous function on it attains a minimum and a maximum.

(d) $f(x, y) = 0$ is a perfectly fine example.

**Problem 2.** The curve parametrized by $r(t) = \langle t, \sqrt{\frac{3}{2}}t^2, t^3\rangle, -\infty < t < \infty$ is a variant of the twisted cubic.

(a) Find the length of the portion of the twisted cubic that lies inside the sphere $x^2+y^2+z^2 = 7/2$.

(b) When $r(t)$ intersects this sphere, what angle does the tangent vector $r'(t)$ make with the normal vector to the sphere?

(a) $r(t)$ intersects this sphere at $t = \pm 1$, since $(\pm 1)^2 + \frac{3}{2}(\pm 1)^4 + (\pm 1)^6 = \frac{7}{2}$. If you couldn't guess this, then you could try to solve for $t$ by plugging the coordinate functions into the equation for the sphere; you'd get a cubic equation disguised as a sextic equation that factors nicely. Note also that $|t| > 1$ implies $r(t)$ lies outside the sphere and $|t| < 1$ implies that $r(t)$ lies inside the sphere, since $\|r(t)\|$ is increasing when $|t|$ is increasing. We have $r'(t) = \langle 1, \sqrt{6}t, 3t^2\rangle$. Therefore, the desired length we need to compute is

$$\begin{aligned} L &= \int_{-1}^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \int_{-1}^{1} \sqrt{1+6t^2+9t^4}dt \\ &= \int_{-1}^{1} \sqrt{(1+3t^2)^2}dt \\ &= \int_{-1}^{1} |1+3t^2|dt \\ &= \int_{-1}^{1} (1+3t^2)dt. \end{aligned}$$

(Note that $1+3t^2$ is always positive.) This ends up evaluating to $L = 4$.

(b) Again, $r(t)$ intersects the sphere at $t = \pm 1$; let's take the intersection at $t = 1$ (the other one will give a similar answer). Here, $r'(t) = \langle 1, \sqrt{6}, 3$. The normal vector is parallel to $r(1) = \langle 1, \sqrt{3/2}, 1\rangle$, which we can see from computing $\nabla(x^2+y^2+z^2-7/2)$ or by just using the fact that a normal vector to a sphere points radially. The angle between these two is

$$\arccos\left(\frac{\langle 1, \sqrt{3/2}, 1\rangle \cdot \langle 1, \sqrt{6}, 3\rangle}{\|\langle 1, \sqrt{3/2}, 1\rangle\|\|\langle 1, \sqrt{6}, 3\rangle\|}\right) = \arccos\left(\frac{7}{4\sqrt{7/2}}\right) = \arccos\left(\frac{\sqrt{14}}{4}\right)$$

If you chose the normal vector to the sphere to point in the opposite direction or if you chose the $t = -1$ point instead, you might have gotten $\pi$ minus the answer above, which is also acceptable.

**Problem 3.** Find a function $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\nabla f = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z\rangle.$$

$f(x, y, z) = x^3yz - 3xy + 2z^2$ is a solution. If you couldn't see this right away or guess this, you could do this sytematically by writing

$$f(x, y, z) = \int (3x^2yz - 3y)dx = x^3yz - 3xy + g(y, z),$$

where $g(y, z)$ is some function of $y, z$ that is independent of the variable $x$. We need $f_y = x^3z - 3x$, so this must equal $\frac{\partial}{\partial y}(x^3yz - 3xy + g(y, z)) = x^3z - 3x + \frac{\partial}{\partial y}g(y, z)$, so $g(y, z) = h(z)$ must be constant with respect to $y$. Doing something similar with the third coordinate tells you that $h(z) = z^2 + C$ for some constant $C$, so we conclude that $f(x, y, z) = x^3yz - 3yx + 2z^2 + C$ is the general solution.

**Problem 4.** Prove that the following system of equations has at least two solutions $(x, y, \lambda)$:

$$\begin{aligned} ye^{yx} &= 4x^3\lambda \\ xe^{xy} &= 6y^5\lambda \\ x^4 + y^6 &= 2 \end{aligned}$$

(You do not need to find these solutions.)

This system of equations is the system of equations associated to the Lagrange multipliers method of finding the extrema of the function $f(x, y) = e^{xy}$ subject to the constraint $x^4 + y^6 = 2$. Since the curve $x^4 + y^6 = 2$ is closed and bounded (neither coordinate can get large because $x^4$ and $y^6$ are always positive), $f$ takes on both a minimum and a maximum on the constraint. These extrema always give two different solutions to the above system, since the Lagrange multipliers theorem tells us that extrema can only occur where the system has a solution.

**Problem 5.** Determine the point(s) on the ellipsoid $\frac{x^2}{16} + \frac{y^2}{9} + (z-1)^2 = 1$ that is/are furthest from the origin.

We want to maximize the function $\sqrt{x^2 + y^2 + z^2}$ subject to the constraint $g(x, y, z) = \frac{x^2}{16} + \frac{y^2}{9} + (z-1)^2 - 1 = 0$. It's equivalent to maximize the easier function $f(x, y, z) = x^2 + y^2 + z^2$ on this contraint. We have $\nabla f = (2x, 2y, 2z)$ and $\nabla g = (x/8, 2y/9, 2(z-1))$, so we need to solve

$$\begin{aligned} 2x &= \lambda x/8 \\ 2y &= 2\lambda y/9 \\ 2z &= 2\lambda(z-1) \\ \frac{x^2}{16} + \frac{y^2}{9} + (z-1)^2 &= 1 \end{aligned}$$

The first equation tells us that either $x = 0$ or $\lambda = 16$. We split into these two cases:

- $x = 0$: The second equation tells us that either $y = 0$ too or $\lambda = 9$. If $x = y = 0$, then we must have $z = 0$ or 2 in order to satisfy the constratint, so the points $(0, 0, 0)$ and $(0, 0, 2)$ are candidates; the distance from the first point to the origin is 0, and the distance of the second is 2. If $x = 0, \lambda = 9$, then the third equation tells us that $z = 9/8$; the corresponding $y$ values are $\pm\frac{9\sqrt{7}}{8}$, so $(0, \pm\frac{9\sqrt{7}}{8}, 9/8)$ are candidates. Their lengths are both $9/\sqrt{8}$, which is a little more than 3. (You should be able to estimate this even without a calculator.)
- $\lambda = 16$: the second equation tells us $y = 0$. The last equation tells us $z = 16/15$, which solves to $x = \pm 4\sqrt{224}/15$. Therefore, our final candidates are the two points $(\pm 4\sqrt{224}/15, 0, 16/15)$. The distance from both of these points to the origin is $16/\sqrt{15}$, which is a little more than 4.

This exhausts all possibilities, so we conclude that the furthest points are $(\pm 4\sqrt{224}/15, 0, 16/15)$ at distance $16/\sqrt{15}$ from the origin.