Worksheet #18: Midterm 2 Review, Part 1
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Instructor: Norman Sheu
Section Leader: CJ Dowd

I am ill at these numbers.

—Polonius, *Hamlet*, Act II, Scene 2.

Problem 1.

(a) Determine whether $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{(x+1)(x^2+xy+y^2)}$ exists, and if it does, find its value.

(b) Describe the domain of the function $f(x,y) = \frac{\sqrt{xy}}{x}$. Determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, and if it does, find its value.

(a) We can factor $x^3 - y^3 = (x-y)(x^2+xy+y^2)$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{(x+1)(x^2+xy+y^2)} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(x^2+xy+y^2)}{(x+1)(x^2+xy+y^2)} = \lim_{(x,y)\to(0,0)} \frac{x-y}{x+1}.$$

Now this function is continuous at $(x,y) = (0,0)$, so we can just evaluate it at this point, which gives 0. Therefore the limit exists and is equal to 0.

(b) The domain of this function is the first and third quadrants, excluding the x-axis but including the y-axis minus the origin. The limit does not exists, since along the path $y = 0$ the limit is 0 but along the path $x = y$ in the first quadrant the limit is 1.

Problem 2. Describe the critical points of $f(x,y,z) = e^{-(xyz)^2}$. Determine if this function has a global minimum; if so, find it and describe where it is attained. Do the same for a global maximum.

We have $\nabla f(x,y,z) = e^{-(xyz)}(-2xy^2z^2, -2x^2yz^2, -2x^2y^2z)$. This is the zero vector if and only if one of the variables is 0, so we conclude that the set of critical points is the union of the xy-, xz-, and yz-planes.

At all of these critical points, we have $f(x,y,z) = e^0 = 1$. This is a maximum of f since e raised to any nonpositive number is at most 1. Since all the critical points gives global maxima, there is no global minimum, since it would have to be attained at a critical point. (f approaches 0 as the variables go to infinity, but never actually attains the value 0.)

Problem 3. Suppose that a function $f(x,y,z)$ of three variables only depends on the length of its input; that is, there exists some function $g : \mathbb{R}^{\geq 0} \to \mathbb{R}$ such that $f(x,y,z) = g(\|r\|)$ where $\vec{r} = (x,y,z)$. Such f is called a *radial* function.

Find an expression for the gradient $\nabla f(x,y,z)$ in terms of g and r. This means that your final answer should not include any of the symbols f, x, y, or z.

This is an application of the chain rule. We have

$$\begin{aligned}
\nabla f(x,y,z) &= \left(\frac{\partial}{\partial x} f(x,y,z), \frac{\partial}{\partial y} f(x,y,z), \frac{\partial}{\partial z} f(x,y,z)\right) \\
&= \left(\frac{\partial}{\partial x} g(\|r\|), \frac{\partial}{\partial y} g(\|r\|), \frac{\partial}{\partial z} g(\|r\|)\right)
\end{aligned}$$

By the chain rule, we have

$$\frac{\partial}{\partial x} g(\|r\|) = g'(\|r\|)\frac{\partial \|r\|}{\partial x} = g'(r)\frac{x}{\sqrt{x^2+y^2+z^2}},$$

and similarly for the y and z partials, so that

$$\begin{aligned}\nabla f(x,y,z) &= \left(g'(\|r\|)\frac{x}{\sqrt{x^2+y^2+z^2}}, g'(\|r\|)\frac{y}{\sqrt{x^2+y^2+z^2}}, g'(\|r\|)\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\\ &= g'(\|r\|)\frac{r}{\|r\|}.\end{aligned}$$

Problem 4. A woman is climbing a very steep slope, described by the plane $f(x,y) = y$. To make the climb easier, she decides to walk it in a switchback. That is, instead of taking the steepest possible ascent, which result in an incline of 45 degrees from going in the direction $(0,1)$, she walks in some oblique direction $\vec{u} = (a,b)$ for a shallower angle of incline. There are two directions in the (x,y) plane that she can walk to give an incline of 10 degrees. What are these directions?

The directional derivative $D_{\vec{u}}f$ gives the slope m from going in the direction $\vec{u}$ when $\vec{u}$ is a unit vector. The slope m is related to the angle of incline θ by $m = \tan(\theta)$. Therefore, we are looking for a unit vector (a,b) such that

$$D_{(a,b)}f = \tan(10°)$$

Note that $\nabla f = (0,1)$, so using the formula for the directional derivative $D_{\vec{u}}f = u \cdot \nabla f$ we need

$$(a,b)\cdot(0,1) = \tan(10°)$$

so $b = \tan(10°)$. The requirement that (a,b) is a unit vector tells us that $a = \pm\sqrt{1-\tan(10°)^2}$. Therefore, the two directions are $(\pm\sqrt{1-\tan(10°)^2}, \tan(10°))$.

Problem 5. Find the minimum and maximum values of $f(x,y) = 8x^2 - 2y$ subject to the constraint $x^2+y^2 \leq 1$ and state where they are attained.

Note that this region is closed and bounded, so by the extreme value theorem f must attain a minimum and a maximum on this region, which is the unit ball.

We first look for critical points on the interior of this ball. But $\nabla f = (16x, -2, 0)$, so there are no critical points. Therefore any extremum must be attained on the boundary $x^2+y^2 = 1$, and we can solve this using Lagrange multipliers.

Letting $g(x,y) = x^2+y^2-1$, we have $\nabla g = (2x, 2y)$. Therefore Lagrange multipliers tells us that any minimum or maximum satisfies

$$\begin{aligned}16x &= \lambda(2x)\\ -2 &= \lambda(2y)\end{aligned}$$

for some constant λ. The first equation tells us that $x = 0$ or $\lambda = 8$, so we consider both cases. If $x = 0$, then we get the points $(0,\pm 1)$ on the sphere. If $\lambda = 8$, then the second equation tells us $y = -1/8$, which gives the points $(\pm\sqrt{63/64}, -1/8)$ on the sphere. These are all the Lagrange points, so there's nothing left to do except check the value of f at all of these and see which one is biggest and which one is smallest.

$$f(\pm\sqrt{63/64}, -1/8) = 8(63/64) - 2(-1/8) = \frac{65}{8} \text{ (for both } \pm) f(0,1) = -2$$
$$f(0,-1) = 2.$$

Therefore the maximum is $\frac{65}{8}$, which occurs at the two points $(\sqrt{63/64}, -1/8)$ and $(-\sqrt{63/64}, -1/8)$. The minimum is -2, which occurs at $(0, 1)$.