Worksheet #17: Lagranger Things
Date: 10/12/2022
Math 53: Fall 2022
Instructor: Norman Sheu
Section Leader: CJ Dowd

Problem 1.

(a) If $g(x, y, z)$ is a smooth function of two variables and $g(P) = 0$, why is $\nabla g(P)$ normal to the surface defined by $g(x, y, z) = 0$? What happens if $\nabla g(P)$ is the zero vector? If you're having trouble visualizing this, try the example of the cone $g(x, y, z) = x^2 + y^2 - z^2$ at $P = (0, 0, 0)$.

(b) Assume $\nabla g(P) \neq 0$. If P is a local maximum of another function $f(x, y, z)$ when constrained to the surface $g(x, y, z) = 0$, why must $\nabla f(P)$ be normal to the surface (and therefore parallel to $\nabla g(P)$)?

(a) If $\vec{u}$ is tangent to the surface at P, then travelling in the $\vec{u}$ direction does not change $g(x, y, z)$ since the surface is a set where g is constant. That is, $D_{\vec{u}} g(P) = 0$. By the formula for directional derivatives, we have

$$D_{\vec{u}} g(P) = \nabla g(P) \cdot \vec{u} = 0,$$

so $\nabla g(P)$ is perpendicular to $\vec{u}$. This is true for any vector $\vec{u}$ tangent to the surface, so we conclude that $\nabla g(P)$ is normal to the surface at P.

If $\nabla g(P)$ is the zero vector, then $D_{\vec{u}} g(P) = 0$ for any vector $\vec{u}$, not just in the tangent directions. The surface $g(x, y, z) = 0$ often looks like it has a sharp point, aka a singular point, at P when this happens.

(b) If P is a local maximum of $f(x, y, z)$ on $g(x, y, z) = 0$, then there cannot be any direction $\vec{u}$ tangent to the surface at P for which $D_{\vec{u}} f$ is negative, since otherwise travelling in the direction of $\vec{u}$ would decrease f while remaining on the surface. Decomposing $\nabla f(P)$, we can write $\nabla f(P) = \lambda \nabla g(P) + \vec{u}$, where $\lambda = \operatorname{comp}_{\nabla g(P)} \nabla f(P)$. The "leftover" vector $\vec{u}$ must be perpendicular to $\nabla g(P)$, so the directional derivative in the $\vec{u}$ direction is

$$D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} = (\lambda \nabla g(P) + \vec{u}) \cdot \vec{u} = \|\vec{u}\|^2.$$

If $\vec{u}$ is nonzero, then this directional derivative is positive. Since $\vec{u}$ lies on the tangent plane to P, traveling in the $\vec{u}$ direction stays on the surface $g(x, y, z) = 0$, so we conclude that traveling in the $\vec{u}$ direction increases f while staying on the surface. Therefore, if P is truly a maximum, we must have $\vec{u} = 0$, or equivalently $\nabla f(P) = \lambda g(P)$. This proves the statement of Lagrange multipliers.

Problem 2. Consider the function $f(x, y) = xy$ on the constraint $2x + y = 3$.

(a) Use Lagrange multipliers to find candidates for local extrema.

(b) Alternatively, by substituting $y = 3 - 2x$, treat $f(x, y) = f(x, 3 - 2x)$ as a single variable function in x when subjected to the constraint. Find its critical points. You should get the same points as part (a).

(c) Does f have a global minimum subject to the constraint? What about a global maximum?

(a) The constraint function is $g(x, y) = 2x + y - 3 = 0$. We have $\nabla f = (y, x)$ and $\nabla g = (2, 1)$. Lagrange multipliers gives the system of equations

$$\begin{aligned} y &= 2\lambda \\ x &= \lambda \end{aligned}$$

Subbing this back into the constraint gives

$$2\lambda + 2\lambda - 3 = 0$$

which implies $\lambda = 3/4$, and thus $x = 3/4, y = 3/2$. This is the only Lagrange point, and therefore the only candidate for a maximum or minimum of f under the constraint.

(b) We have $f(x, 3 - 2x) = x(3 - 2x) = -2x^2 + 3x$. The single-variable derivative is $-4x + 3$, so setting this to zero the only critical point is $x = 3/4$. We can backsubstitute to get $y = 3/2$.

(c) From part (b), it should be clear that the point $(3/4, 3/2)$ gives a global maximum from what you know about single variable calculus. (I don't think this is obvious from the Lagrange multipliers version.) However, there is no global minimum: the line $2x + y - 3 = 0$ is unbounded, and $f(x, y)$ becomes arbitrarily negative as you go off in either direction to ∞.

Problem 3. (Multiple constraints.) Lagrange multipliers can be generalized to the scenario when the objective function is subject to more than one constraint. If $f(x, y, z)$ attains an extreme value at a point P when constrained to both $g(x, y, z) = 0$ and $h(x, y, z) = 0$, then there exists constants $\lambda, \mu \in \mathbb{R}$ such that

$$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P).$$

In the language of linear algebra, this is saying that $\nabla f(P)$ is a linear combination of the other two gradients.

The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse. Find the highest and lowest points on this ellipse, i.e. the points with minimum and maximum z-values.

The objective function is $f(x, y, z) = z$, and our constraints are $g(x, y, z) = 4x - 3y + 8z - 5$ and $h(x, y, z) = x^2 + y^2 - z^2$. We have

$$\begin{aligned} \nabla f &= (0, 0, 1) \\ \nabla g &= (4, -3, 8) \\ \nabla h &= (2x, 2y, -2z), \end{aligned}$$

so Lagrane multipliers gives us the system

$$\begin{aligned} 0 &= 4\lambda + 2\mu x \\ 0 &= -3\lambda + 2\mu y \\ 1 &= 8\lambda - 2\mu z, \end{aligned}$$

along with the constraints. The first two equations imply $\mu(6x + 8y) = 0$, so either $\mu = 0$ or $3x + 4y = 0$. However, if $\mu = 0$, then $\lambda = 0$ and the last equation is false, so we conclude that any solution must have $3x + 4y = 0$. Plugging this into the constraints, this is

$$\begin{aligned} 4x - 3(3x/4) + 8z - 5 &= 0 \\ x^2 + (3x/4)^2 + z^2 &= 0 \end{aligned}$$

and finally solving this gives two solutions $(x, y, z) = (-4/3, 1, 5/3)$ and $(4/13, -3/13, 5/13)$. Since there are only two Lagrange points and the region is closed and bounded, one of them must be the maximum and the other must be the minimum, so we conclude that $(-4/3, 1, 5/3)$ is the highest point and $(4/13, -3/13, 5/13)$ is the lowest point (since $5/3 > 5/13$).