Worksheet #17: Lagranger Things
Date: 10/12/2022
Math 53: Fall 2022
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Problem 1.

(a) If $g(x, y, z)$ is a smooth function of two variables and $g(P) = 0$, why is $\nabla g(P)$ normal to the surface defined by $g(x, y, z) = 0$? What happens if $\nabla g(P)$ is the zero vector? If you're having trouble visualizing this, try the example of the cone $g(x, y, z) = x^2 + y^2 - z^2$ at $P = (0, 0, 0)$.

(b) Assume $\nabla g(P) \neq 0$. If P is a local maximum of another function $f(x, y, z)$ when constrained to the surface $g(x, y, z) = 0$, why must $\nabla f(P)$ be normal to the surface (and therefore parallel to $\nabla g(P)$)?

Problem 2. Consider the function $f(x, y) = xy$ on the constraint $2x + y = 3$.

(a) Use Lagrange multipliers to find candidates for local extrema.

(b) Alternatively, by substituting $y = 3 - 2x$, treat $f(x, y) = f(x, 3 - 2x)$ as a single variable function in x when subjected to the constraint. Find its critical points. You should get the same points as part (a).

(c) Does f have a global minimum subject to the constraint? What about a global maximum?

Problem 3. (Multiple constraints.) Lagrange multipliers can be generalized to the scenario when the objective function is subject to more than one constraint. If $f(x, y, z)$ attains an extreme value at a point P when constrained to both $g(x, y, z) = 0$ and $h(x, y, z) = 0$, then there exists constants $\lambda, \mu \in \mathbb{R}$ such that

$$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P).$$

In the language of linear algebra, this is saying that $\nabla f(P)$ is a linear combination of the other two gradients.

The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse. Find the highest and lowest points on this ellipse, i.e. the points with minimum and maximum z-values.