Worksheet #16: Going Supercritical
Date: 10/10/2022
Math 53: Fall 2022
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Problem 1. Let $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function of two variables. True or false:

1. Every local extremum (minimum or maximum) of f occurs at a critical point.
2. If f has a critical point at P and $f_{xx}(P)$, $f_{yy}(P)$, and $f_{xy}(P)$ are all positive, then f has a local minimum at P.
3. If f has a local maximum at P, then $f_{xx}(P)$ and $f_{yy}(P)$ must both be nonpositive.

1. True. If f has a local minimum at $P = (a,b)$, then in particular the single variable function $f(x,b)$ has a local minimum at $x = a$. Since f is differentiable, by single variable calculus this means that we must have $\frac{\partial f}{\partial x} = 0$ at P. Likewise $\frac{\partial f}{\partial y} = 0$, and the same argument holds if we consider a local maximum. Therefore we must have $\nabla f = (0,0)$ at P in either case.
2. False. If f_{xy} is sufficiently large compared to f_{xx}, f_{yy}, i.e. when $f_{xy}(P) > \sqrt{f_{xx}(P)f_{yy}(P)}$, then $D = f_{xx}f_{yy} - (f_{xy})^2$ will be negative and so f will have a saddle point at P. An example of a function where this happens is $f(x,y) = x^2 + y^2 + 3xy$.
3. True. Again, we look at the single variable case: if f has a local maximum at $P = (a,b)$, then the single variable function $f(x,b)$ also has a local maximum at P, so by the single variable version of second derivative test we must have $f_{xx}(P) \le 0$. The same holds for the y-direction. (Note that it is possible for f_{xx} and f_{yy} to be zero; for example, the constant function $f(x,y) = 0$ has every point as a local maximum.)

Problem 2. Let $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ be the function that measures distance from the origin. Compute $D_{\vec{u}}f(1,2,3)$ for the three following values of $\vec{u}$:

$$\frac{1}{\sqrt{14}}\langle 1,2,3\rangle,\ \langle -1,-1,1\rangle,\ \text{and}\ \langle 1,2,3\rangle + \langle -1,-1,1\rangle.$$

We have $\nabla f = \frac{1}{\sqrt{x^2+y^2+z^2}}\langle x,y,z\rangle$, so $\nabla f(1,2,3) = \frac{1}{\sqrt{14}}\langle 1,2,3\rangle$. Using the formula $D_{\vec{u}}f = \nabla f \cdot \vec{u}$ for directional derivatives, we compute

$$D_{\frac{1}{\sqrt{14}}\langle 1,2,3\rangle} f(1,2,3) = (\frac{1}{\sqrt{14}}\langle 1,2,3\rangle) \cdot (\frac{1}{\sqrt{14}}\langle 1,2,3\rangle) = \|\frac{1}{\sqrt{14}}\langle 1,2,3\rangle\|^2 = 1$$
$$D_{\langle -1,-1,1\rangle} f(1,2,3) = (\frac{1}{\sqrt{14}}\langle 1,2,3\rangle) \cdot \langle -1,-1,1\rangle = 0$$
$$D_{\langle 1,2,3\rangle + \langle -1,-1,1\rangle} = \sqrt{14} D_{\frac{1}{\sqrt{14}}\langle 1,2,3\rangle} f(1,2,3) + D_{\langle -1,-1,1\rangle} f(1,2,3) = \sqrt{14}.$$

For this last one, we saved some work by using the linear properties of the directional derivative to reuse the results of the previous two directional derivatives.

Problem 3. (Stewart Exercise 14.7.60.) Find an equation

$$a(x-1) + b(y-2) + c(z-3) = 0$$

of the plane that contains the point $(1,2,3)$ and cuts the smallest possible volume off of the corner of the first octant.

- You may assume without proof that the minimum volume is actually attained by some plane, i.e. there is an answer to this problem.
- It may be helpful to know that the volume V of a simplex (a four-sided pyramid whose faces are all triangles) with vertices $(0,0,0),(x,0,0),(0,y,0)$, and $(0,0,z)$ is $V = \frac{xyz}{6}$.
- To check your answer, the correct minimum volume is 27.
- I think this quite a difficult problem, though it is doable with the tools available to you from lecture. Do it only if you've finished everything else. If you want to fully finish the problem within a reasonable amount of computation, then you'll need to be smart about simplifying where you can.

Note that a, b, and c must all be positive, otherwise the volume is infinite. The intersection of the plane with the x-axis is the value of x for which $a(x-1)-2b-3c=0$, which is $x = \frac{2b+3c}{a}+1$. Similarly, the y- and z-axis intercepts are $y = \frac{a+3c}{b}+2$ and $z = \frac{a+2b}{c}+3$. By the volume formula, we want to minimize the following function as a function of a, b, c:

$$\begin{aligned} f(a,b,c) &= \frac{1}{6}\left(\frac{2b+3c}{a}+1\right)\left(\frac{a+3c}{b}+2\right)\left(\frac{a+2b}{c}+3\right) \\ &= \frac{1}{6}\left(\frac{a+2b+3c}{a}\right)\left(\frac{a+2b+3c}{b}\right)\left(\frac{a+2b+3c}{c}\right) \\ &= \frac{(a+2b+3c)^3}{6abc}. \end{aligned}$$

There are at least two ways to optimize this function:

- The non-calculus competition math way: use the AM-GM inequality. The AM-GM inequality tells us that $\frac{(a+2b+3c)^3}{27} \geq 6abc$ for any positive numbers a, b, c, with equality if and only if $a = 2b = 3c$. The equality case is the one that minimizes the expression $\frac{(a+2b+3c)^3}{6abc}$, and rescaling the vector (a, b, c) gives the same plane, so we could take $a = 6, b = 3, c = 2$.

  I don't expect you to understand this solution or have any idea what the AM-GM inequality is. But for those of you who are familiar with it, this is probably the most straightforward reason why the minimum volume is given by this point.

- Using multivariable calculus: using the quotient rule, we are solving for

  $$\begin{aligned}(0,0,0) = \nabla f = \frac{1}{6}\frac{1}{(abc)^2}(&3abc(a+2b+3c)^2 - bc(a+2b+3c)^3, 6abc(a+2b+3c)^2 - ac(a+2b+3c)^3, \\ &9abc(a+2b+3c)^2 - ab(a+2b+3c)^3).\end{aligned}$$

  This might look horrible, but remember we're only trying to solve for $\nabla f = 0$. this means that we can cancel any nonzero factor each coordinate, which includes anything that looks like a product of a, b, and c and terms of the form $(a+2b+3c)$. After this simplication, we reduce to solving the system of equations

  $$\begin{aligned} 3a &= a+2b+3c \\ 6b &= a+2b+3c \\ 9c &= a+2b+3c. \end{aligned}$$

But in particular, this tells us that $3a = 6b = 9c$, so the vector (a, b, c) is determined up to scalar multiplication to be $(6, 3, 2)$. (Again, rescaling does not affect the plane or the value of f.) This vector and its scalar multiples are the only critical points of f, so if f acheives a (global) minimum it must do so here, since global minima are also local minima and every local minimum is a critical point. Since its implied by problem statement, I'm allowing you to assume that a global minimum does exist, so we're done.