Worksheet #15: You Have Nothing to Lose but Your Chain Rule
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Math 53: Fall 2022
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Problem 1. Let $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function of three variables. If we take the level set $f(x, y, z) = 0$, this makes z into an implicit function of x and y (at least at most points).

(a) Recall the implicit differentiation formulas

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}, \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$$

and the definition of the gradient

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, and ∇f for

$$f(x, y, z) = yz + x \ln y - z^2.$$

(b) For your results in part (a), show that ∇f is perpendicular to both $(1, 0, \frac{\partial z}{\partial x})$ and $(0, 1, \frac{\partial z}{\partial y})$ (at least when all three of these things are defined).

(c) Show more generally that part (b) is true for any function f assuming that the partial derivatives involved exist and are finite. Give a conceptual explanation for why this should be true. (Hint: first explain why the vectors $(1, 0, \frac{\partial z}{\partial x})$ and $(0, 1, \frac{\partial z}{\partial y})$ both lie in the tangent plane to the surface defined by $f(x, y, z) = 0$. How can you relate this to the gradient?)

(a)

$$\frac{\partial z}{\partial x} = -\frac{\ln y}{y - 2z}$$
$$\frac{\partial z}{\partial y} = -\frac{z + x/y}{y - 2z}$$
$$\nabla f = (\ln y, z + x/y, y - 2z)$$

(b)

$$(1, 0, \frac{\partial z}{\partial x}) \cdot \nabla f = (1, 0, -\frac{\ln y}{y - 2z}) \cdot (\ln y, z + x/y, y - 2z) = \ln y + 0 - \ln y = 0$$
$$(0, 1, \frac{\partial z}{\partial y}) \cdot \nabla f = (0, 1, -\frac{z + x/y}{y - 2z}) \cdot (\ln y, z + x/y, y - 2z) = 0 + z + x/y - (z + y/x) = 0.$$

(c) In general,

$$(1, 0, \frac{\partial z}{\partial x}) \cdot \nabla f = (1, 0, -\frac{\partial f/\partial x}{\partial f/\partial z}) \cdot (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z) = \partial f/\partial x - \partial f/\partial x = 0,$$

and similarly

$$(0,1,\frac{\partial z}{\partial y})\cdot\nabla f=(0,1,-\frac{\partial f/\partial y}{\partial f/\partial z})\cdot(\partial f/\partial x,\partial f/\partial y,\partial f/\partial z)=\partial f/\partial y-\partial f/\partial y=0,$$

both assuming $\partial f/\partial z\neq 0$. In short, the reason why this is true is that ∇f is normal to the surface $f(x,y,z)=0$, whereas both $(1,0,\partial z/\partial x)$ and $(1,0,\partial z/\partial y)$ lie in the tangent plane to the surface since they run along the level set $f(x,y,z)=0$. Therefore, they must be orthogonal.

Problem 2. (Stewart Exercise 14.5.55.) A function f is called *homogeneous of degree n* if it satisfies the equation

$$f(tx,ty)=t^n f(x,y)$$

for any $t\in\mathbb{R}$, where n is some positive integer.

(a) Show that $f(x,y)=x^2y+2xy^2+5y^3$ is homogeneous of degree 3. (The multivariate homogeneous polynomials are probably the most common source of homogeneous functions.)

(b) Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y).$$

(Hint: Use the chain rule to differentiate both sides of $f(tx,ty)=t^nf(x,y)$ with respect to t.)

(a)

$$\begin{aligned}f(tx,ty)&=(tx)^2(ty)+2(tx)(ty)^2+5(ty^3)\\&=t^3x^2y+2t^3xy^2+5t^3y^3\\&=t^3(x^2y+2xy^2+5y^3)\\&=t^3f(x,y).\end{aligned}$$

(b) Differentiating both sides of $f(tx,ty)=t^nf(x,y)$ with respect to t gives

$$\begin{aligned}\frac{d}{dt}(f(tx,ty))&=\frac{d}{dt}(t^nf(x,y))\\f_x(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+f_y(tx,ty)\cdot\frac{\partial(ty)}{\partial t}&=nt^{n-1}f(x,y)\\x\cdot f_x(tx,ty)+y\cdot f_y(tx,ty)&=nt^{n-1}f(x,y).\end{aligned}$$

This equation is true for any value of t, so in particular it is true for $t=1$, whence

$$x\cdot f_x(x,y)+y\cdot f_y(x,y)=nf(x,y).$$