Worksheet #15: You Have Nothing to Lose but Your Chain Rule
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Math 53: Fall 2022
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Problem 1. Let $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function of three variables. If we take the level set $f(x, y, z) = 0$, this makes z into an implicit function of x and y (at least at most points).

(a) Recall the implicit differentiation formulas

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}, \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$$

and the definition of the gradient

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, and ∇f for

$$f(x, y, z) = yz + x \ln y - z^2.$$

(b) For your results in part (a), show that ∇f is perpendicular to both $(1, 0, \frac{\partial z}{\partial x})$ and $(0, 1, \frac{\partial z}{\partial y})$ (at least when all three of these things are defined).

(c) Show more generally that part (b) is true for any function f assuming that the partial derivatives involved exist and are finite. Give a conceptual explanation for why this should be true. (Hint: first explain why the vectors $(1, 0, \frac{\partial z}{\partial x})$ and $(0, 1, \frac{\partial z}{\partial y})$ both lie in the tangent plane to the surface defined by $f(x, y, z) = 0$. How can you relate this to the gradient?)

Problem 2. (Stewart Exercise 14.5.55.) A function f is called *homogeneous of degree* n if it satisfies the equation

$$f(tx, ty) = t^n f(x, y)$$

for any $t \in \mathbb{R}$, where n is some positive integer.

(a) Show that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3. (The multivariate homogeneous polynomials are probably the most common source of homogeneous functions.)

(b) Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

(Hint: Use the chain rule to differentiate both sides of $f(tx, ty) = t^n f(x, y)$ with respect to t.)