Worksheet #14: Impartiality
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Math 53: Fall 2022
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Problem 0. Let $f(x, y)$ and $g(u, v)$ be two functions related by

$$g(u, v) = f(e^u + \sin v, e^u + \cos v).$$

Use the following values to calculate $g_u(0, 0)$ and $g_v(0, 0)$ (not all of these will be relevant):

$$\begin{array}{llll} f(0,0) = 3 & g(0,0) = 6 & f_x(0,0) = 4 & f_y(0,0) = 8 \\ f(1,2) = 6 & g(1,2) = 3 & f_x(1,2) = 2 & f_y(1,2) = 5. \end{array}$$

(Hint: this is an application of the multivariate chain rule. How is $g_u = \frac{\partial}{\partial u} f(e^u \sin v, e^u + \cos v)$ related to f_x and f_y?)

(Solution is on the previous worksheet.)

Problem 1.

(a) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth[1] functions. If $f_x = g_x$, what can you say about the difference $f - g$? What if you replace x with y? If both $f_x = g_x$ and $f_y = g_y$, what can you say about $f - g$?

(b) Find a function $f(x, y)$ such that $f_x = 2xy + 4$ and $f_y = x^2 - 12y^3$.

(c) Prove that there does not exist a function $f(x, y)$ with $f_x = x$ and $f_y = x$.

(d) (Bonus.) Generalize parts (b) and (c) in the following ways. Given two smooth functions $g, h : \mathbb{R}^2 \to \mathbb{R}$, give a necessary and sufficient condition for the existence of f such that $f_x = g, f_y = h$. Describe a procedure/algorithm to compute f (up to a constant) given f_x, f_y. Generalize to an arbitrary number of input variables. Be as rigorous as you can about justifying your steps.

(a) Since $\frac{\partial}{\partial x}(f - g) = f_x - g_x$, if $f_x = g_x$ then we must have $\frac{\partial}{\partial x}(f - g) = 0$. The only functions whose derivatives are 0 are constant functions; in the setting of partial derivative setting, this means that the function is constant with respect to the variable that is being differentiated. Therefore we conclude that $f - g$ is constant with respect to x, or equivalently that it is a function $\alpha(y)$ that only depends on y.

Similarly, if $f_y = g_y$, then we conclude that f and g differ by a function $\beta(x)$ that is constant with respect to y. If both $f_x = g_x$ and $f_y = g_y$, then f and g must differ by a function that is simultaneously constant with respect to x and with respect to y, so the difference is some (genuine) constant C.

(b) The general solution is $f(x, y) = x^2y + 4x - 3y^4 + C$ for some constant C. To get this, we can integrate $f_x(x, y)$ with respect to x to determine f up to a function of y. (When we integrate a multivariable function, we treat the variables not being integrated against as if they were

[1]This means that the partial derivatives of all orders exist; that is, $\frac{\partial^{i+j} f}{\partial x^i \partial y^j}$ exists for any pair of integers $i, j \geq 0$. It's usually nice to assume smoothness for a function you don't know much about since then you can differentiate without worrying.

constant, just like we treat the variables we are not differentiating as constants when taking partial derivatives.)

$$f(x,y) = \int f_x(x,y)dx = \int (2xy+4)dx = x^2y + 4x + \alpha(y),$$

where $\alpha(y)$ is some unknown function of y. To determine $\alpha(y)$, and thus determine f, we differentiate with respect to y and compare to the known expression for f_y:

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}(x^2y + 4x + \alpha(y)) \\ x^2 - 12y^3 &= x^2 + \alpha'(y). \end{aligned}$$

Therefore $\alpha'(y) = -12y^3$, so $\alpha(y) = -3y^4 + C$, and thus $f(x,y) = x^2y + 4x - 3y^4 + C$. Since adding a constant to f neither affects f_x nor f_y, this is the most specific we can get for set of solutions.

(c) If such f existed, then we have $f_{xy} = \frac{\partial}{\partial y}(f_x) = 0$ but $f_{yx} = \frac{\partial}{\partial x}(f_y) = 1$. This is a contradiction since we must have $f_{xy} = f_{yx}$ by Clairaut's theorem.

Problem 2. (Stewart 14.3, Exercises 88 & 89.) Consider n moles of gas sitting in a container of volume V at pressure P and temperature T. The ideal gas law is the equation $PV = nRT$, where R is some constant. This means that given three of the variables P, V, n, T, you can determine the fourth.

(a) Show that $\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = -1$. (To do this, you'll need to write P as a function of the other variables, then V as a function of the other variables, and then T as a function of the other variables.)

(b) Show that $T \cdot \frac{\partial P}{\partial T} \cdot \frac{\partial V}{\partial T} = nR$.

(a) We can write each of the three variables P, V, T as functions of the others:

$$\begin{aligned} P &= \frac{nRT}{V} \\ V &= \frac{nRT}{P} \\ T &= \frac{PV}{nR}. \end{aligned}$$

Using these equations, we compute

$$\begin{aligned} \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} &= \left(-\frac{nRT}{V^2}\right)\left(\frac{nR}{P}\right)\left(\frac{V}{nR}\right) \\ &= -\frac{nRT}{PV} \\ &= -1 \end{aligned}$$

where $\frac{nRT}{PV} = 1$ comes from rearranging the ideal gas law.

(b) Again using the functions from part (a), we have

$$\begin{aligned} T \cdot \frac{\partial P}{\partial T} \cdot \frac{\partial V}{\partial T} &= T \left(\frac{nR}{V} \right) \left(\frac{nR}{P} \right) \\ &= nR \left(\frac{nRT}{PV} \right) \\ &= nR. \end{aligned}$$