Worksheet #14: Impartiality
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Math 53: Fall 2022
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Problem 0. Let $f(x, y)$ and $g(u, v)$ be two functions related by

$$g(u, v) = f(e^u + \sin v, e^u + \cos v).$$

Use the following values to calculate $g_u(0, 0)$ and $g_v(0, 0)$ (not all of these will be relevant):

$$f(0,0) = 3 \qquad g(0,0) = 6 \qquad f_x(0,0) = 4 \qquad f_y(0,0) = 8$$
$$f(1,2) = 6 \qquad g(1,2) = 3 \qquad f_x(1,2) = 2 \qquad f_y(1,2) = 5.$$

(Hint: this is an application of the multivariate chain rule. How is $g_u = \frac{\partial}{\partial u} f(e^u \sin v, e^u + \cos v)$ related to f_x and f_y?)

Problem 1.

(a) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth[1] functions. If $f_x = g_x$, what can you say about the difference $f - g$? What if you replace x with y? If both $f_x = g_x$ and $f_y = g_x$, what can you say about $f - g$?

(b) Find a function $f(x, y)$ such that $f_x = 2xy + 4$ and $f_y = x^2 - 12y^3$.

(c) Prove that there does not exist a function $f(x, y)$ with $f_x = x$ and $f_y = x$.

(d) (Bonus.) Generalize parts (b) and (c) in the following ways. Given two smooth functions $g, h : \mathbb{R}^2 \to \mathbb{R}$, give a necessary and sufficient condition for the existence of f such that $f_x = g, f_y = h$. Describe a procedure/algorithm to compute f (up to a constant) given f_x, f_y. Generalize to an arbitrary number of input variables. Be as rigorous as you can about justifying your steps.

Problem 2. (Stewart 14.3, Exercises 88 & 89.) Consider n moles of gas sitting in a container of volume V at pressure P and temperature T. The ideal gas law is the equation $PV = nRT$, where R is some constant. This means that given three of the variables P, V, n, T, you can determine the fourth.

(a) Show that $\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = -1$. (To do this, you'll need to write P as a function of the other variables, then V as a function of the other variables, and then T as a function of the other variables.)

(b) Show that $T \cdot \frac{\partial P}{\partial T} \cdot \frac{\partial V}{\partial T} = mR$.

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[1]This means that the partial derivatives of all orders exist; that is, $\frac{\partial^{i+j} f}{\partial x^i \partial y^j}$ exists for any pair of integers $i, j \geq 0$. It's usually nice to assume smoothness for a function you don't know much about since then you can differentiate without worrying.