Worksheet #13: Partial Credit
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Problem 1. What goes wrong if you try to generalize the definition

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

to define the derivative of a function f of more than one variable?[1]

The natural way to want to do this is to define the total derivative of f to be

$$\lim_{(h,k)\to(0,0)} \frac{f(x+h,y+k)-f(x,y)}{(h,k)} = f'(x,y).$$

Remember, we know how to take limits of functions of more than one variable. However, there is an obvious problem with the above: we're dividing by a vector, which is ridiculous. The way this is fixed is by defining $f'(x,y)$ to be a *vector*, not a number, associated to each point (x,y) in the domain of f, which approximates f linearly (i.e. giving a tangent plane) by

$$f(x+h,y+k) \approx f'(x,y)\cdot(h,k) + f(x,y).$$

In this case, $f'(x,y)$ will be the vector of partial derivatives (f_x, f_y), also known as the *gradient* of f, also denoted ∇f. We will discuss gradients in much more detail later.

There are a number of notations for a partial derivative. If $f(x,y)$ is a function of two variables, then f_x, $\frac{\partial f}{\partial x}$ and D_x all mean the same thing. You should get used to all of them.

Problem 2. Let $f(x,y)$ and $g(u,v)$ be two functions related by

$$g(u,v) = f(e^u + \sin v, e^u + \cos v).$$

Use the following values to calculate $g_u(0,0)$ and $g_v(0,0)$ (not all of these will be relevant):

$f(0,0)=3$	$g(0,0)=6$	$f_x(0,0)=4$	$f_y(0,0)=8$
$f(1,2)=6$	$g(1,2)=3$	$f_x(1,2)=2$	$f_y(1,2)=5.$

(Hint: this is an application of the multivariate chain rule. How is $g_u = \frac{\partial}{\partial u} f(e^u \sin v, e^u + \cos v)$ related to f_x and f_y?)

In this instance, the chain rule takes the form

$$\begin{aligned}
g_u &= \frac{\partial}{\partial u} f(e^u+\sin v, e^u+\cos v)\\
&= f_x(e^u+\sin v, e^u+\cos v)\cdot\frac{\partial}{\partial u}(e^u+\sin v) + f_y(e^u+\sin v, e^u+\cos v)\cdot\frac{\partial}{\partial u}(e^u+\cos v)\\
&= f_x(e^u+\sin v, e^u+\cos v)\cdot e^u + f_y(e^u+\sin v, e^u+\cos v)\cdot e^u.
\end{aligned}$$

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[1] There is a good way to think about *the* derivative of a multivariable function using a slight modification of this definition, usually called the *total derivative* to distinguish it from the partials. In the case $f:\mathbb{R}^n\to\mathbb{R}$, the total derivative the same thing as the gradient.

Substituting $(u, v) = (0, 0)$ gives

$$\begin{aligned} g_u(0,0) &= f_x(1,2) \cdot 1 + f_y(1,2) \cdot 1 \\ &= 2 + 5 = 7 \end{aligned}$$

using given table of values for $f_x(1,2)$ and $f_y(1,2)$. Similarly, differentiating with respect to v instead of u, we have

$$g_v = f_x(e^u + \sin v, e^u + \cos v) \cdot (\cos v) + f_y(e^u + \sin v, e^u + \cos v) \cdot (-\sin v)$$

and thus

$$g_v(0,0) = f_x(1,2) \cdot 1 + f_y(1,2) \cdot 0 = 2 + 0 = 2.$$

Problem 3. Compute $f_{yxyxx}(2,0)$ if $f(x,y) = \frac{7}{8}(y^2 + y + x)^4$. (The notation f_{yxyxx} means the same thing as $\frac{\partial^5 f}{\partial x^3 \partial y^2}$. Remember that you can choose to do the partial derivatives in any order, and some orders are easier than others.)

We'll compute f_{xxxyy} instead, since by Clairaut's theorem changing the order in which we take the partial derivatives does not matter. We have

$$\begin{aligned} f_x &= \frac{7}{2}(y^2 + y + x)^3 \\ f_{xx} &= \frac{21}{2}(y^2 + y + x)^2 \\ f_{xxx} &= 21(y^2 + y + x) \\ f_{xxxy} &= 42y + 21 \\ f_{xxxyy} &= 42. \end{aligned}$$

Therefore $f_{yxyxx} = f_{xxxyy}$ is the constant function 42, so in particular $f_{yxyxx}(2,0) = 42$. (If you want to see Clairaut's theorem in action, try doing the partials in a different order. It could get quite messy, but there will eventually be a "miraculous" cancellation/simplification that gives the same answer as before.)