Worksheet #12: Limit Break
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Problem 1.

(a) Explain the difference between these two statements:

(i) The limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

(ii) The limits of $f(x,y)$ along any line through (a,b) all exist and are equal.

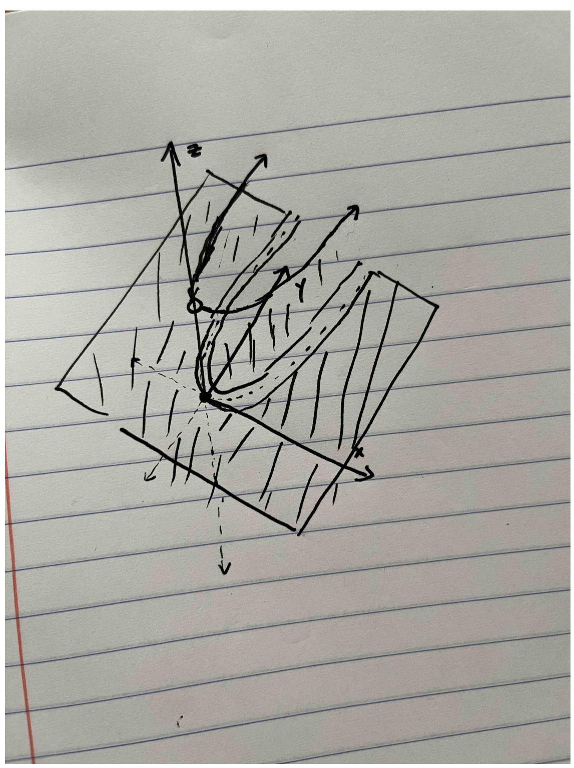
(b) Sketch a 3D graph of the piecewise defined function

$$f(x,y) = \begin{cases} 1: & \text{if } x^2 = y \text{ and } x \neq 0 \\ 0: & \text{else.} \end{cases}$$

(This might feel very weird.) Setting $(a,b) = (0,0)$, show that statement (i) is false for f but statement (ii) is true.

(a) Let the domain of f be D. The limit of a function $f(x,y)$ at (a,b) exists if the limits of $f(r(t))$ exist and are all equal for all paths $r(t) : \mathbb{R} \to D$ that pass through (a,b). Statement (ii) weakens this statement by only requiring us to consider paths that are straight *lines*, but in general there are many other paths.

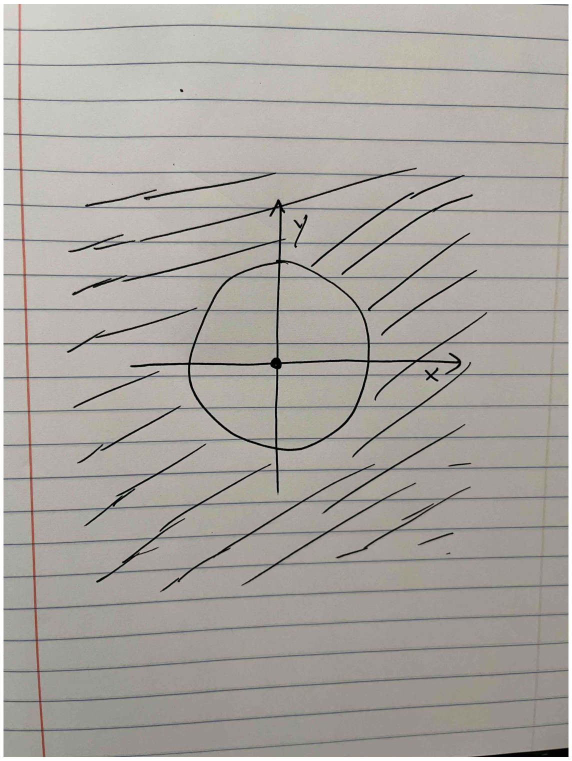
(b) Geometrically, the graph of this function looks mostly like the xy-plane, but with a parabola-shaped wire removed from it and lifted upwards by 1, except with $(0,0)$ left in place. Here's my attempt at a drawing:



The limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist, since a path that goes to $(0,0)$ along the parabola is constant at $f(x,y)=1$ but most other paths are eventually constant at $f(x,y)=0$. In particular, f is eventually constant at 0 along any linear path to $(0,0)$, since a line in $\mathbb{R}^2$ through the origin intersects the parabola $x^2 = y$ in at most one point other than $(0,0)$, and after this other intersection point, f is constant at 0 along the line.

Problem 2. Suppose f is a function whose domain is $D = \{(0,0)\} \cup \{(x,y) : |x^2+y^2| \geq 1\}$. (Ask me or your groupmates about this notation if it is new to you.) Sketch the domain D in the plane. Is f continuous at $(0,0)$?[1]

This is the set of points in the plane lying outside the unit circle (inclusive) as well as the lone point $(0,0)$:



Regardless of the other properties of f, it must be continuous at the origin in a sort of vacuous sense. Let's see this in two ways using both the path definition of a limit and the epsilon-delta definition:

(a) Paths: The only continuous path $\gamma : \mathbb{R} \to D$ on the domain D of f that passes through is the constant path $\gamma(t) = (0,0)$. Since this is literally the only path we can take that goes to $(0,0)$, we conclude that the limit of f along every path going to $(0,0)$ exists and is equal to $f(0,0)$, so f is continuous at $(0,0)$.

(b) Epsilon-delta[2]: The epsilon-delta definition of continuity states that $f(x,y)$ is continuous at $(0,0)$ if, given any fixed $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x,y) - f(0,0)| < \epsilon$ whenever $(x,y) \in D$ satisfies $\|(x,y) - (0,0)\| < \delta$.

[1] You can do this without using any other information about f. If this exercise confuses you, I'd suggest looking at Definition 1 in Stewart 14.2 very carefully.

[2] If this makes sense to you, great! Otherwise, don't worry too much since the epsilon-delta definition of a limit is not something we will emphasize in this course.

In the case at hand, given any $\epsilon > 0$ (it doesn't matter how small it is), choose $\delta = 1/2$. Then the *only* point $(x, y) \in D$ with $\|(x, y) - (0, 0)\| < 1/2$ is $(x, y) = (0, 0)$ itself, and certainly $|f(0, 0) - f(0, 0)| = 0 < \epsilon$. Therefore the definition of continuity is satisfied.

Problem 3. Suppose that S is a (sufficiently nice and smooth) surface in $\mathbb{R}^3$ that contains the two curves

$$r_1(t) = (2 + 3t, 1 - t^2, 3 - 4t^2)$$
$$r_2(u) = (1 + u^2, 2u^3 - 1, 2u + 1).$$

Using this information, compute an equation of the tangent plane to S at the point $(2, 1, 3)$. (What do you have to do to find the normal vector of this plane?)

Note that $r_1(0) = r_2(1) = (2, 1, 3)$. The tangent vectors $r_1'(0)$ and $r_2'(1)$ both lie in the tangent plane at $(2, 1, 3)$, so to find the normal vector of this plane we just need to take their cross product. We have

$$r_1'(0) = (3, 0, 0)$$
$$r_2'(1) = (2, 6, 2)$$
$$r_1'(0) \times r_2'(1) = (0, -6, 18).$$

Since $(2, 1, 3)$ is itself on the tangent plane, we can therefore write the plane as

$$-6(y - 1) + 18(z - 3) = 0.$$