Worksheet #11: Art Class
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Math 53: Fall 2022
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Problem 1. Because you will need it: in 3D space, draw…

(a) Labeled coordinate axes

(b) A plane

(c) A sphere

(d) A torus (donut shape)

(e) A helix/spring shape

(f) A paraboloid

(g) A cube

(h) A cylinder (with finite height)

(i) A (single) cone (with finite height)

(j) A pyramid/tetrahedron

(k) An octahedron

The exact dimensions and orientation of the shapes don't matter, but you should try for a genuine 3D appearance. I'd suggest dotting lines that are in the background and choosing your perspective wisely. Shading is also an option. Most imporantly, make it clear that the objects you are drawing could be recognized unambiguously by someone else.

Problem 2. (Based on Stewart 13.4.44) The *angular momentum* of a particle with mass m and position vector $r(t)$ is $L(t) = mr(t) \times v(t)$, and its *torque* is $\tau(t) = mr(t) \times a(t)$.

(a) Note that we treat both of these quantities as vectors; which directions do these vectors point? What is the particle "rotating" around with these formulas?

(b) Prove that $L'(t) = \tau(t)$, and thus conservation of angular momentum: $L(t)$ is constant if $\tau(t) = 0$.

(c) Show that a particle moving in a straight line at a constant *velocity* has constant angular momentum in two ways: i) using (b) and ii) doing the computation directly.

(d) Show that a particle moving in a circle with constant *speed* about the origin also has constant angular momentum using any method you like.

(a) The rotation here is about the origin. You can use the right hand rule to determine the direction of the angular momentum and torque. If the particle's movement is constricted to a plane, then the torque and angular momentum will be vectors that are pointing straight up or down out of the plane. You should think of the angular momentum vector as a rod fixed at the origin that the particle is "curling" around.

(b) This follows from the "product rule" for cross products that we discuss last time:

$$\begin{aligned} L'(t) &= \frac{d}{dt}[mr(t) \times v(t)] \\ &= mr'(t) \times v(t) + mr(t) \times v'(t). \end{aligned}$$

By definition, $r'(t) = v(t)$ and $v'(t) = a(t)$, so this is

$$= mv(t) \times v(t) + mr(t) \times a(t).$$

Any vector crossed with itself gives 0, so we are left with $L'(t) = mr(t) \times a(t) = \tau(t)$.

(c) i) Since acceleration is 0, so is torque. ii) The motion of the particle can be parameterized as $r(t) = P + tv_0$ for some fixed point P and constant velocity vector v_0. Then

$$L(t) = m(P + tv_0) \times v_0 = mP \times v_0$$

using linearity of the cross product and the fact that $v_0 \times v_0 = 0$. Therefore $L(t)$ is independent of t, so it is constant.

(d) At least two ways to do this: if a particle is moving in a plane, say the xy-plane, then the z-component of its acceleration vector must be 0, else the particle would leave the plane. However, since the speed is constant, we showed last time that the acceleration vector must be perpendicular to the velocity vector. Thus $a(t)$ must be perpendicular to both the z-axis and $v(t)$, from which we deduce that $a(t)$ is parallel to the position vector $r(t)$; it points inward (if this isn't clear to you, make a drawing). Therefore $mr(t) \times a(t) = 0$.

Alternatively, you can parameterize circular motion explicitly as $r(t) = r_0(\cos(\omega t), \sin(\omega t), 0)$ where r_0 and ω are constants that control the radius of motion and the angular velocity. Then you can compute directly that $a(t) = -\omega^2 r_0(\cos(\omega t), \sin(\omega t), 0)$, which is parallel to $r(t)$ so again we get zero torque.