Worksheet #10: I'm Running Out of Ideas for Amusing Vector-Themed Worksheet Titles
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Problem 1. A particle starts at position $r(0) = (3, 1, 2)$ and has velocity given by $r'(t) = (2t, 0, e^t)$. What is its position at time $t = 3$?

To get a position vector from a velocity vector, you need to integrate the velocity vector. Therefore,

$$r(t) = \int (2t, 0, e^t)dt = (t^2 + A, 0 + B, e^t + C)$$

where A, B, C are some constants. We have an initial condition $r(0) = (3, 1, 2)$ which we use to solve for A, B, C; we conclude $r(t) = (t^2 + 3, 1, e^t + 1)$, so $r(3) = (12, 1, e^3 + 1)$.

Problem 2. What's the difference between a vector function and a parametric curve?

Not a damn thing. (You should really think of these as the same types of objects: a vector function $r : \mathbb{R} \to \mathbb{R}^n$ is the same as giving n parametric equations $r_1 = r_1(t), r_2 = r_2(t), \ldots, r_n = r_n(t)$. The calculus is the same for both, so in fact you were already doing vector calculus from day 1. You might argue that we only did parametric equations in 2 dimensions, but there's no reason we had to limit ourselves to this other than to make the beginning of the course easier to digest. Like parametric curves, you are welcome to restrict the domain of a vector function to whatever you like—a choice of domain is part of the definition of a function. For example, $r : [0, 2\pi] \to \mathbb{R}^n, r(t) = (\cos t, \sin t, \cos(2t))$ is a perfectly valid vector function.)

Problem 3. Prove that taking derivatives of a cross/dot product of vector functions $u(t), v(t)$ obeys the "product rule":

(a) $\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$

(b) $\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$

Use the the first of these to show that if $r(t)$ has constant speed (explain what this means!), then $r''(t)$ is orthogonal to $r'(t)$.

$$\begin{aligned}
\frac{d}{dt}[u(t) \cdot v(t)] &= \frac{d}{dt}\left[\sum_{i=1}^{n} u_i(t)v_i(t)\right] \\
&= \sum_{i=1}^{n} \left(\frac{d}{dt} u_i(t)v_i(t)\right) \\
&= \sum_{i=1}^{n} \left(u_i'(t)v_i(t) + u_i(t)v'(t)\right) \\
&= \sum_{i=1}^{n}(u_i'(t)v_i(t)) + \sum_{i=1}^{n}(u_i(t)v_i'(t)) \\
&= u'(t) \cdot v(t) + u(t) \cdot v'(t).
\end{aligned}$$

Here, we use the fact that the derivative of a sum is the sum of the derivatives (taking derivatives is *linear*) and the single-variable product rule. The version for the cross product is morally identical but messier to write out explicitly. I'll omit the variable t for conciseness:

$$\begin{aligned}\frac{d}{dt}[u(t)\cdot v(t)] &= \frac{d}{dt}(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\\ &= (u_2'v_3 + u_2v_3' - u_3'v_2 - u_3v_2', u_3'v_1 + u_3v_1' - u_1'v_3 - u_1v_3', u_1'v_2 + u_1v_2' - u_2'v_1 - u_2v_1')\\ &= (u_2'v_3 - u_3'v_2, u_3'v_1 - u_1'v_3, u_1'v_2 - u_2'v_1) + (u_2v_3' - u_3v_2', u_3v_1' - u_1v_3', u_1v_2' - u_2v_1')\\ &= u'(t)\times v(t) + u(t)\times v'(t).\end{aligned}$$

If $r(t)$ has constant speed, this means that $\|r'(t)\| = c$ for some fixed constant c independent of the value of t. Therefore

$$c^2 = \|r'(t)\|^2 = r'(t)\cdot r'(t).$$

(Never forget the utility of writing a length squared as a dot product!) Differentiating both side of this equation gives

$$\begin{aligned}0 &= r''(t)\cdot r'(t) + r'(t)\cdot r''(t)\\ &= 2r'(t)\cdot r''(t)\end{aligned}$$

so $r'(t)\cdot r''(t) = 0$, which means that $r'(t)$ is always orthogonal to $r''(t)$.

Problem 4. For a vector-valued function $r : \mathbb{R} \to \mathbb{R}^3$, give an interpretation of the three vector functions $T(t) = r'(t)/\|r'(t)\|$, $N(t) = T'(t)/\|T'(t)\|$, and $T(t)\times N(t)$. These are called the unit tangent, normal, and binormal vectors respectively. Prove that they are mutually orthogonal and of unit length (we call a set of vectors with this property "orthonormal").

As the name suggests, the unit tangent vector $T(t)$ is the direction the curve $r(t)$ is pointing at time t. The unit normal vector $N(t)$ points in the direction that the curve is *turning*. You can think of it as a steering wheel. The unit normal vector will always be perpendicular to the unit tangent vector by the last part of Problem 3. The unit binormal vector is the *axis about which this turn occurs*; think of it as a pole in space that the curve is wrapping around.

The unit normal vector will always be perpendicular to the unit tangent vector by the last part of Problem 3—the unit tangent vector T has constant length, so its derivative T' is orthogonal to it, and rescaling T' does not change its direction. Since the unit binormal vector is the cross product of the other two, we conclude that it is perpendicular to both; therefore the three vectors are mutually orthogonal. By definition, $T(t)$ and $N(t)$ have length 1. Since T and N are orthogonal (i.e. the angle between them is $\pi/2$), their cross product has length $\|T(t)\|\|N(t)\|\sin(\theta) = 1$, so we conclude that the binormal vector also has length 1.