Worksheet #9: Midterm 1 Review
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These questions are somewhat more difficult than what I would expect of the real midterm. You will also have more time per question on the actual midterm. If you can answer all of these with only minor algebra mistakes, you should be in excellent shape.

Problem 1. Write down a polar equation for the line $y = \sqrt{2}(1-x)$ of the form $r = f(\theta)$. (Draw a graph of the line. Try rotating to get something you know.)

I misprinted this problem; I originally intended it to be $y = \frac{\sqrt{2}}{2} - x$. Here is the solution to the intended version of this problem: rotating this line 45 degrees clockwise gives the vertical line $x = 1$, which has polar equations $r = \sec\theta$. We can rotate this line back 45 degrees counterclockwise by replacing θ with $\theta - \pi/4$ to get $r = \sec(\theta - \pi/4)$ as a polar equation for the given line. Alternatively, you can instead start by rotating 45 degrees counterclockwise to get the line $y = 1$, which has polar equations $r = \csc\theta$, so the given line has polar equations $r = \csc(\theta + \pi/4)$. (You can see that this is equivalent to the previous answer by using the trig identity $\cos(\theta) = \sin(\theta + \pi/2)$.)

Now here is the solution to the printed version of the problem $y = \sqrt{2}(1-x)$. The same general strategey works, but instead we have to rotate clockwise by $\arctan(1/\sqrt{2})$ to get a vertical line $x = a$. The value a will be the distance from the origin to the line, which stays constant under this rotation. Using triangles, we can compute this distance to be $a = \sqrt{\frac{2}{3}}$. Applying the same rotation transformation, we conclude that a polar equation is

$$r = \sqrt{\frac{2}{3}}\sec(x - \arctan(1/\sqrt{2})).$$

Despite what I originally claimed in the hint, you can also do this by subbing $x = r\cos\theta, y = r\sin\theta$, and it isn't even that hard (oops). You get

$$\begin{aligned} r\sin\theta &= \sqrt{2}(1 - r\cos\theta) \\ r(\sin\theta + \sqrt{2}\cos\theta) &= \sqrt{2} \\ r &= \frac{\sqrt{2}}{\sin\theta + \sqrt{2}\cos\theta}. \end{aligned}$$

If you really want to, you can check that this is the same as the previous answer using the angle addition formula.

Problem 2. What is the rightmost point of the curve defined by parametric equations $x = 2t - t^2, y = (t-1)^3, -\infty < t < \infty$? What is equation of the tangent line at this point? (Bonus: What does the slope of the tangent line tell you about whether the graph is "smooth" or "pointy" here?)

The rightmost point occurs where $x = 2t - t^2$ is maximized. We solve for this by setting $\frac{dx}{dt} = 2 - 2t$ to 0; the only zero is at $t = 1$, which corresponds to $x = 1, y = 0$.

The formula for the tangent line is

$$\frac{dy/dt}{dx/dt} = \frac{3(t-1)^2}{2(t-1)}.$$

At $t = 1$, this is indeterminate, but we can take the limit to cancel a factor of $t - 1$ and conclude that the tangent line has slope 0. Therefore the equation of the tangent line is $y = 0$.

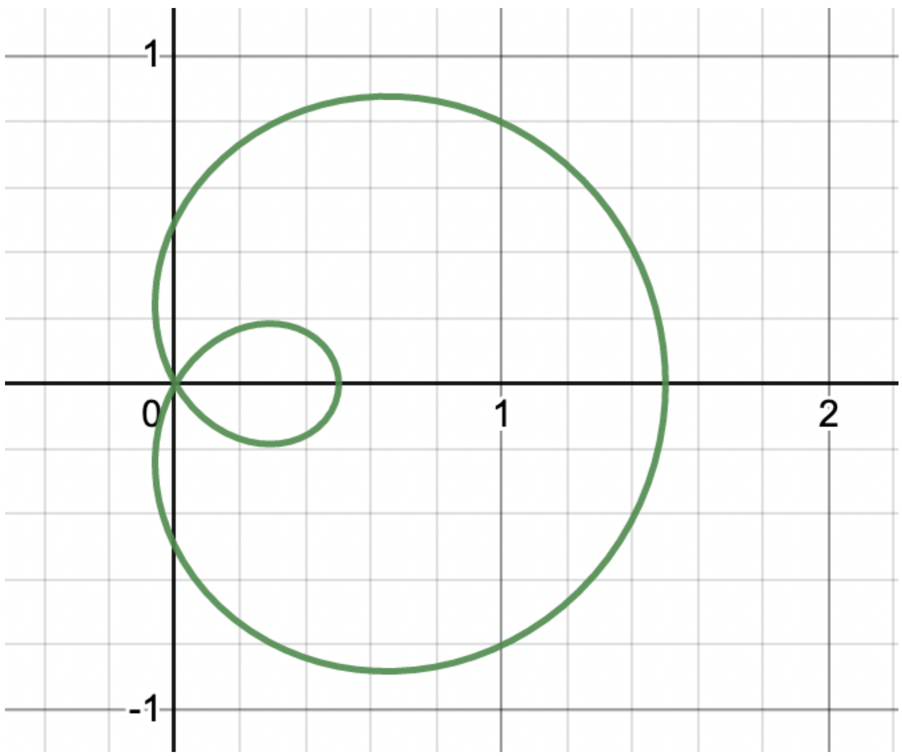
If the curve were smooth at $(1, 0)$, then the tangent would be vertical since the value of x is maximized here. But the tangent is horizontal, so the curve must have some type of cusp at $(1, 0)$.

Problem 3. Sketch the limaçon $r = \frac{1}{2} + \cos\theta$ and find the area between the outer loop and the inner loop.

Tips on sketching: good points to plot are

$$(3/2, 0), (1/2, \pi/2), (0, 2\pi/3), (-1/2, \pi), (0, 4\pi/3), (1/2, 3\pi/2), (3/2, 2\pi).$$

Connecting the dots between these points in order should give you a reasonable looking graph and make it clear how to picutre the inner and outer loop. Be aware of the intervals on which the radius is descreasing and increasing, and be especially mindful of the interval with negative radius, which gives the inner loop. The graph looks like this:



The desired area can be computed as

$$A = \frac{1}{2}\int_0^{2\pi} (1/2 + \cos\theta)^2 d\theta - 2\left(\frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1/2 + \cos\theta)^2 dx\right).$$

The first integral computes the entire area inside the limaçon but double-counts the area in the small loop. The second integral computes the area in the small loop (the bounds are the two values of θ for which $r = 0$). We subtract two times this from the first integral to get just the area between the two loops.

We have

$$\begin{aligned}\int (1/2+\cos\theta)^2 d\theta &= \int (1/4+\cos\theta+\cos^2\theta)d\theta \\ &= \int (1/4+\cos\theta+\frac{1}{2}+\frac{1}{2}\cos 2\theta)d\theta \\ &= \frac{3}{4}\theta+\sin\theta+\frac{1}{4}\sin 2\theta + C\end{aligned}$$

so applying this to the definite integrals above gives

$$\begin{aligned}A &= \frac{1}{2}\left[\frac{3}{4}\theta+\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi} - \left[\frac{3}{4}\theta+\sin\theta+\frac{1}{4}\sin 2\theta\right]_{2\pi/3}^{4\pi/3} \\ &= \frac{1}{2}\left[\frac{3\pi}{2}+0+0\right] - \left[\frac{\pi}{2}-\sqrt{3}+\frac{\sqrt{3}}{4}\right] \\ &= \frac{\pi}{4}+\frac{3\sqrt{3}}{4}.\end{aligned}$$

Problem 4.

(a) Find the angle between a diagonal of a cube and one of its edges.

(b) Find a vector perpendicular to both a diagonal and a side of the cube.

(a) We can define a cube in $\mathbb{R}^3$ by having vertices at the 8 points of the form (a, b, c) where a, b, c are either 0 or 1. One diagonal of this cube is the vector $(1, 1, 1)$, which points from $(0, 0, 0)$ to $(1, 1, 1)$, and we can take any side coming out of the origin, say $(1, 0, 0)$. Using the angle formula, the angle between the diagonal and the side is

$$\begin{aligned}\theta &= \arccos\left(\frac{(1,1,1)\cdot(1,0,0)}{\|(1,1,1)\|\|(1,0,0)\|}\right) \\ &= \arccos\left(\frac{1}{\sqrt{3}}\right).\end{aligned}$$

(If you made different choices of diagaonals or side, e.g. if you flipped one of the vectors, you might get $\arccos\left(-\frac{1}{\sqrt{3}}\right)$, which is also acceptable.)

(b) We can find such a vector by taking the cross product between a diagonal vector and a side vector:

$$(1, 1, 1) \times (1, 0, 0) = (0 - 0, 1 - 0, 0 - 1) = (0, 1, -1).$$

Again, other answers are possible depending on your choice of diagonal and side, and the length of the answer doesn't matter. All correct answers should look vaguely like the one given above, with one coordinate 0 and the other coordinates the same up to sign.

Problem 5. Show that the planes $2x - 3y + z = 4$ and $4x - 6y + 2z = 3$ are parallel and find the distance between them.

These planes are parallel since their normal vectors $(2, -3, 1)$ and $(4, -6, 2)$ are scalar multiples of each other. To compute the distance between the planes, take any point on the first plane, say $P_1 = (0, 0, 4)$, and find the closest point on the other plane by taking the normal line through P_1, which is parameterized as $(0, 0, 4) + t(2, -3, 1)$, and finding the point P_2 where it intersects the second plane. We solve

$$\begin{aligned} 4(2t) - 6(-3t) + 2(4 + t) &= 3 \\ 28t &= -5 \end{aligned}$$

so the vector between P_1 and P_2 is $-\frac{5}{28}(2, -3, 1)$. The length of this vector is

$$\sqrt{\left(-\frac{10}{28}\right)^2 + \left(\frac{15}{28}\right)^2 + \left(-\frac{5}{28}\right)^2} = \frac{\sqrt{350}}{28}.$$

so this is the distance between the planes.