Worksheet #8: Crucifixus
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Problem 1. Are there any vectors $v \in \mathbb{R}^3$ such that $(1, 2, 1) \times v = (3, 1, -5)$? If so, find all of them; otherwise, prove that none exist. Do the same question for $(1, 2, 1) \times v = (3, 1, 5)$.

For the second question, the answer is no becauase $(3, 1, 5)$ is not perpendicular to $(1, 2, 1)$, and the cross product $\vec{a} \times \vec{b}$ is always perpendicular to both $\vec{a}$ and $\vec{b}$. We can tell that these two vectors are not orthogonal since their dot product $(3, 1, 5) \cdot (1, 2, 1) = 10$ is not zero.

However, $(3, 1, -5)$ *is* orthogonal to $(1, 2, 1)$ so we should expect some solutions in this case. Here are two ways to do this:

1. Solution 1: Set up a linear system. Let $v = (a, b, c)$. Then

$$(1, 2, 1) \times v = (2c - b, a - c, b - 2a),$$

so we can solve for v by setting this cross product to $(3, 1, -5)$. This gives us a linear system of 3 equations in 3 variables:

$$\begin{aligned} 2c - b &= 3 \\ a - c &= 1 \\ b - 2a &= -5. \end{aligned}$$

You can use any method you like to do this; I like substitution. We have $c = a - 1$, so the system becomes

$$\begin{aligned} 2(a - 1) - b &= 3 \\ b - 2a &= -5 \end{aligned}$$

and both of these are equivalent to $2a - b = 5$, or $b = 2a - 5$. We can let a take on any value, and then b and c are functions of a, so our complete solution set is

$$(a, 2a - 5, a - 1), a \in \mathbb{R}.$$

We can verify quickly that all vectors of this form give the right cross product:

$$(1, 2, 1) \times (a, 2a - 5, a - 1) = (2(a - 1) - (2a - 5), a - (a - 1), (2a - 5) - 2a) = (3, 1, -5).$$

2. Solution 2: Find one solution and then add multiples of $(1, 2, 1)$. If we find one solution v_0 to the equation, then

$$v_0 + t(1, 2, 1)$$

is also a solution, since

$$(1, 2, 1) \times (v_0 + t(1, 2, 1)) = (1, 2, 1) \times v_0 + (1, 2, 1) \times t(1, 2, 1) = (3, 1, -5) + (0, 0, 0)$$

since the cross product is linear and the cross product of any two parallel vectors is 0. Every solution must be of the form $v_0 + t(1, 2, 1)$, since if u is not a scalar multiple of $(1, 2, 1)$ then

$$(1, 2, 1) \times (v_0 + u) = (3, 1, -5) + (1, 2, 1) \times u$$

where $(1, 2, 1) \times u$ is nonzero, so $(1, 2, 1) \times (v_0 + u)$ cannot be equal to $(3, 1, -5)$.

Therefore we only need to find one solution v_0 and we get the rest for free. Any nonzero vector perpendicular to $(3, 1, -5)$ will have a scalar multiple that is a solution, so let's find a vector perpendicular to $(3, 1, -5)$ by taking its cross product with something, say $(1, 0, 0)$. (In section, I chose to take its cross product with $(1, 2, 1)$, but really any choice works as long as it's not a multiple of $(3, 1, -5)$).

$$(3, 1, -5) \times (1, 0, 0) = (0, -5, -1).$$

To see how we need to scale this, we check

$$(1, 2, 1) \times (0, -5, -1) = (-2 + 5, 0 + 1, -5) = (3, 1, -5).$$

In this case, we happened to be lucky and chose a vector that was exactly the right length to give the correct cross product, so we can set $v_0 = (0, -5, -1)$. (If you weren't lucky, you'd have to rescale this vector appropriately). Therefore the general solution is

$$(0, -5, -1) + t(1, 2, 1) = (t, 2t - 5, t - 1).$$

Lo and behold, this is exactly the same solution from the first part.

Problem 2. Suppose I am riding a monorail in a straight line in the direction $\vec{v}$. There is a wind blowing with constant force $\vec{F}$ on the monorail. The monorail is sturdy; pushing it in a direction perpendicular to the rail does nothing. With this in mind, what is the *effective* force on the monorail, i.e. the component of the wind force that will actually do anything? How is this situation related to projection and the dot product? What does it have to do with work, in the physics sense? (This type of scenario will become very important not too long from now when we talk about path integrals.)

The effective component of the force on the monorail is the component that is pointing in the same direction as the rail. In other words, the effective force is the *projection* of $\vec{F}$ onto $\vec{v}$, which is $(\vec{F} \cdot \hat{v})\hat{v}$, or $\|\vec{F}\| \cos\theta \hat{v}$ where θ is the angle the force vector makes with $\vec{v}$. This is the the work per distance that the wind does on the monorail as it travels—only the component of force in the direction of travel contributes to work. This is how one does problems involving work when the force isn't pointing in the same direction as motion.

Problem 3. Let L_1 be the line passing through the points $(1, -2, 4)$ and $(2, 1, 3)$, and let L_2 be the line passing through $(0, 3, -3)$ and $(2, 4, 1)$.

(a) Write parametric equations for each of these lines.

(b) Are L_1 and L_2 parallel, skew, or intersecting? If they intersect, where? If they do not intersect, how far apart are they, and where are they closest, i.e. which pair of points $P_1 \in L_1$ and $P_2 \in L_2$ minimizes the distance $d(P_1, P_2)$?

(a) The direction vector for L_1 is

$$v_1 = (2-1, 1-(-2), 3-4) = (1, 3, -1)$$

and the direction vector for L_2 is

$$v_2 = (2-0, 4-3, 1-(-3)) = (2, 1, 4).$$

Therefore, one set of parametric equations for these two lines is

$$L_1 = \{(1,-2,4) + tv_1 = (1+t, -2+3t, 4-t) \mid t \in \mathbb{R}\}$$
$$L_2 = \{(0,3,-3) + tv_2 = (2t, 3+t, -3+4t) \mid t \in \mathbb{R}\}.$$

(b) We can see that L_1 and L_2 are not parallel since their direction vectors are not scalar multiples of each other. Instead of determining whether they are intersecting or skew beforehand, I'm just going to compute the minimum distance between two points on these lines; if this distance is 0, then the lines intersect, and otherwise they are skew.

One method to find the distance between the two lines is to find the component of the vector between any two given points, say $(1,-2,4) \in L_1, (0,3,-3) \in L_2$, that is simulteneously perpendicular to v_1 and v_2. This is

$$\text{comp}_{(1,3,-1)\times(2,1,4)}((1,-2,4) - (0,3,-3) = \text{comp}_{(13,-6,-5)}(1,-5,7) = 8/\sqrt{230}.$$

This is probably the easiest way in general to compute the distance between two lines. However, this doesn't give us the points that minimize the distance.

Actually finding the points was probably too messy of a question to ask for section. Here's one solution: compute the minimum distance of a given point on L_1 to L_2, and then minimize this distance among all points on L_1. We can do both of these steps using single variables calculus, or we can do them at the same time with multivariable calculus. One neat trick to simplify this computation is that minimizing the distance is the same as minimizing the square of the distance, so we can drop all the square roots in our computations. Therefore you are trying to minimize

$$(2t_2 - t_1 - 1)^2 + (t_2 - 3t_1 + 5)^2 + (4t_2 + t_1 - 7)^2.$$

If you want to do this yourself (I'd probably reccomend using a calculator), the correct answer is $t_1 = 233/115, t_2 = 148/115$, and the corresponding points are

$$P_1 = \left(\frac{348}{115}, \frac{469}{115}, \frac{227}{115}\right)$$
$$P_2 = \left(\frac{296}{115}, \frac{493}{115}, \frac{247}{115}\right).$$

and you can check that the two corresponding points lie at distance $8/\sqrt{230}$.