Worksheet #7: Fisiks
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Problem 1. In 3D space, I shoot a cannonball out of a cannon at the speed of light ($= 3 \times 10^8\ m/s$) from the origin. I can describe the direction the cannon is pointing using two angles: the angle θ describes which direction in the xy-plane the cannon is pointing, and the angle φ describes the angle of elevation of the cannon off the ground.

(a) Draw (i) bird's eye view of the cannon in the xy-plane, labeling the angle θ and (ii) a diagram of the cannon from the side, labeling the angle φ.

(b) What is the velocity vector of the cannonball the moment it gets shot? Check your answer by making sure the length of this vector is equal to the speed of light.

(c) Acceleration due to gravity is $10\ m/s^2$. Write down the acceleration vector of the cannonball, ignoring air resistance. Then write down the velocity vector as a function of time t after the cannon gets shot, and then finally write down the position vector of the cannonball as a function of t. (You can assume that t is small enough that the cannonball hasn't hit the ground yet).

(a)

(b) The speed $c = 3 \times 10^8$ of the cannon ball is the length of the velocity vector $\vec{v}_0$. We first break the velocity into its vertical and horizontal components. Letting $\hat{h}$ be the horizontal direction of travel (of unit length) and taking $\vec{k}$ to be the height direction, using a right triangle we conclude that $\vec{v}_0 = c \sin\varphi \vec{k} + c \cos\varphi \hat{h}$. However, we still need to break $\hat{h}$ into its $\vec{i}$ and $\vec{j}$ components; again using a right triangle, the horizontal direction decomposes as $\hat{h} = \cos\theta \vec{i} + \sin\theta \vec{j}$. Putting this all together, we conclude

$$\vec{v}_0 = (c \cos\theta \cos\varphi)\vec{i} + (c \sin\theta \cos\varphi)\vec{j} + (c \sin\varphi)\vec{k}.$$

(c) The acceleration vector is $\vec{a} = (0, 0, -10)$. Integrating and setting initial velocity to $\vec{v}_0$ gives velocity

$$\vec{v} = (c \cos\theta \cos\varphi, c \sin\theta \cos\varphi, c \sin\varphi - 10t)$$

as a function of time. Integrating one more time and setting initial position to 0 finally yields

$$\vec{d} = (tc \cos\theta \cos\varphi, t \sin\theta \cos\varphi, tc \sin\varphi - 5t^2).$$

Problem 2. (Expressing reflection with vectors.)

(a) A beam of light travelling with velocity v bounces off a flat mirror with normal vector n. We can write

$$v = \frac{v \cdot n}{\|n\|^2} n + \left(v - \frac{v \cdot n}{\|n\|^2} n \right).$$

Explain why this is the decomposition of v into its *normal* and *tangent components* relative to the mirror. (If the concept of *projecting* a vector onto another vector is new to you, talk it over with your groupmates!)

(b) Use this decomposition and the fact that reflection is a *linear transformation*[1] to show that the new velocity of the light after reflecting is

$$v - 2\frac{v \cdot n}{\|n\|^2} n.$$

Check to make sure that the speed hasn't changed, i.e. the length of this vector is equal to $\|v\|$.

(c) On the Euclidean plane, a beam of light travels in the vertical straight line $x = x_0$ for some $-1 < x_0 < 1$, going northward at a speed of 1. However, the unit circle has been replaced by a perfectly circular mirror. The beam of light hits this circular mirror and reflects off in a new direction. Draw a picture of the situation and then find the velocity vector of the light beam after reflecting off the mirror; this velocity vector will depend on x_0. Sanity check your answer in the cases $x_0 = 0, \pm\sqrt{2}/2$ and in the limits as $x_0 \to \pm 1$.

(a) The formula is writing v as $\operatorname{proj}_n(v) + (v - \operatorname{proj}_n(v))$. Here $\operatorname{proj}_n v$ is the component of v that is parallel to n, and $v - \operatorname{proj}_n v$ is whatever is leftover, which must be perpendicular to n, i.e. tangent to the mirror. (I'm using Stewart's notation for projection here, but I think it's terrible, since $\operatorname{proj}_n(v)$ is a scalar multiple of n, not v. Having it look like a multiple of v is incredibly misleading.)

(b) If I shine a beam of light directly into a mirror at a 90 degree angle, its direction will simply reverse. Therefore, since $\operatorname{proj}_n v$ is in the normal direction, its direction after reflecting will be $-\operatorname{proj}_n v$. Meanwhile, the tangent component is unaffected by reflection—if you shine a light tangent to a mirror, you can think of it as not hitting the mirror at all. Therefore $v - \operatorname{proj}_n(v)$ is not changed by the reflection. Adding the effects on the two components together using the fact that reflection is a linear transformation, we conclude that the new velocity is $-\operatorname{proj}_n v + v - \operatorname{proj}_n(v)$, which is the same as $v - 2\frac{v \cdot n}{\|n\|^2} n$.

(c) Note that the beam will hit the mirror from the bottom, so the point of impact is $(x_0, -\sqrt{1 - x_0^2})$. The normal vector to the circle points in the radial direction, so it has the same coordinates as this point: $n = (x_0, -\sqrt{1 - x_0^2})$. Note also that $\|n\| = 1$ already. The direction of the light beam before impact is $(0, 1)$, so using the formula from part (b) we compute

$$\begin{aligned}
v - 2\frac{v \cdot n}{\|n\|^2} n &= (0,1) - 2\left[(0,t) \cdot \left(x_0, -\sqrt{1 - x_0^2}\right)\right]\left(x_0, -\sqrt{1 - x_0^2}\right) \\
&= (0,1) + 2\sqrt{1 - x_0^2}\left(x_0, -\sqrt{1 - x_0^2}\right) \\
&= \left(2x_0\sqrt{1 - x_0^2}, -1 + 2x_0^2\right).
\end{aligned}$$

so the new direction is $\left(2x_0\sqrt{1 - x_0^2}, -1 + 2x_0^2\right)$.

We sanity check this: if $x_0 = 0$, then the beam will hit the center of the circle and bounce directly backwards with new velocity $(0, -1)$. If $x_0 = \pm\sqrt{2}/2$, then we expect the light beam to bounce at a right angle from its old direction, and indeed when we plug in these values to our solution we get $(\pm 1, 0)$. Finally, as $x_0 \to \pm 1$, the beam gets closer and closer to being tangent to the mirror, so the limit of the reflection as $x_0 \to \pm 1$ should be $(0, 1)$, as if no reflection had occured. And we do find that setting $x_0 = \pm 1$ gives solution $(0, 1)$.

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[1] A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a function that takes a vector as input and gives a vector as output and satisfies the property $T(\lambda v + \mu w) = \lambda T(v) + \mu T(w)$ for any vectors $v, w \in \mathbb{R}^n$ and scalars λ, μ.