Worksheet #6: Victor Was My Nerd Name
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Problem 1. (Discussion.) In your own words, what is a vector, and what are its properties?[1]

There are a number of ways you can think of a (real) n-dimensional vector.

- As an n-tuple of real numbers, i.e. a list of numbers $v = (v_1, v_2, v_3, \dots, v_n)$.
- As a sum of components of the coordinate axes, e.g. $v = a\vec{i} + b\vec{j} + c\vec{k}$ in 3 dimensions.
- As a point in n-dimensional space.
- As an arrow of a certain length pointing in a given direction in n-dimensional space.
- As something with magnitude and direction: the magnitude of v is $\|v\|$, and the direction could be thought of as $\hat{v}$.

Often it will be best to have 2D or 3D pictures when thinking about vectors; you might be able to extrapolate what's going on in higher dimensions.

The basic operations on vectors are a) vector addition and b) scalar multiplication. Both of these are done *coordinatewise*, i.e. the sum of two vectors is

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

and scaling by a real number λ is

$$\lambda \cdot (x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n).$$

Other concepts associated to a vector $v = (x_1, \dots, x_n)$ are its *length*, or *magnitude*

$$\|v\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

and its normalization, i.e. the unique vector of length 1 pointing in the same direction as v:

$$\hat{v} = \frac{v}{\|v\|}$$

(assuming v is not the zero vector $(0, 0, \dots, 0)$).

You will see many different notations for vectors. When writing a vector v abstractly without writing down coordinates, I like to just use a plain variable, understanding that I'm talking about a vector and not a real number from contect. To help to distinguish the vectors from the scalars, $\vec{v}$ (with an arrow), $\overrightarrow{v}$, and $\mathbf{v}$ are also used, although writing in bold font by hand tends to be difficult. When writing in coordinate notation, the following are all common:

- Row vector notation $(x_1, x_2, \dots, x_n)$, $\langle x_1, \dots, x_n \rangle$, or $[x_1, \dots, x_n]$, either with parentheses, angle brackets, or square brackets

[1] If you respond, "a vector is an element of a vector space," you will be punished severely.

- Column vector notation $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ or $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

- In 3 dimensions, ijk notation: $a\hat{i} + b\hat{j} + c\hat{k}$. Here $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, and $\hat{k} = (0,0,1)$, so that this vector is the same as (a,b,c). Sometimes the hats on i,j,k are dropped.

Column vs. row notation will sometimes matter in more advanced situations, e.g. when we want to use matrix multiplication, but you shouldn't worry about the difference too much now.

Problem 2. If u and v are vectors of lengths 2 and 3 respectively, what are the largest and smallest possible values of $u \cdot v$? Draw pictures of both situations.

When two vectors are pointing in the same direction (i.e. they are positive scalar multiples of each other), their dot product is the product of the lengths and is maximized (in any dimension). Therefore the maximum possible value of $u \cdot v$ is $2 \cdot 3 = 6$.

When two vectors are pointing in opposite direction (i.e. they are negative scalar multiples of each other), their dot product is the negative product of the lengths and is minimized (in any dimension). Therefore the minimum value is -6.

Problem 3. The following are true for vectors $u, v \in \mathbb{R}^3$:

$$u \cdot v = |u||v| \cos\theta$$

$$|u \times v| = |u||v| \sin\theta$$

where θ is the angle between u and v.

- Given u and v, what's the easiest way to compute θ?

- What can you say if $u \cdot v$ is almost as large as or equal to $|u||v|$? When it is almost as negatively large or equal to $-|u||v|$? When it is zero?

- What can you say when $|u \times v|$ is almost as large as or equal to $|u||v| \sin\theta$? When it is zero?

- There are few good reasons to compute the angle using the dot product instead of the cross product. First, the dot product is much simpler to compute. Second, the cross product loses information by having to take the length of $|u \times v|$—it can't tell the difference between θ and $\pi - \theta$. I'd reccommend always computing the angle as
$$\theta = \arccos\left(\frac{u \cdot v}{|u||v|}\right).$$

- $u \cdot v$ is close to $|u||v|$ when u, v are close to pointing in the same direction, with equality when they point in exactly the same direction. $u \cdot v$ is close to $-|u||v|$ when u, v are close to pointing in opposite directions, with equality when they point in exactly the opposite direction. $u \cdot v = 0$ means that u and v are perpendicular.

- $|u \times v| \approx |u||v|$ means that u and v are nearly perpendicular, with equality if they are exactly perpendicular. $u \times v = 0$ means either that u, v point in the same direction or in opposite directions.